

Minimizing Effective Resistance of a Graph*

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Abstract. The effective resistance between two nodes of a weighted graph is the electrical resistance seen between the nodes of a resistor network with branch conductances given by the edge weights. The effective resistance comes up in many applications and fields in addition to electrical network analysis, including, for example, Markov chains and continuous-time averaging networks. In this paper we study the problem of allocating edge weights on a given graph in order to minimize the total effective resistance, i.e., the sum of the resistances between all pairs of nodes. We show that this is a convex optimization problem and can be solved efficiently either numerically or, in some cases, analytically. We show that optimal allocation of the edge weights can reduce the total effective resistance of the graph (compared to uniform weights) by a factor that grows unboundedly with the size of the graph. We show that among all graphs with n nodes, the path has the largest value of optimal total effective resistance and the complete graph has the least.

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1. Introduction. Let \mathcal{N} be a network with n nodes and m edges, i.e., an undirected graph (V, E) with $|V| = n$, $|E| = m$, and nonnegative weights on the edges. We call the weight on edge l its *conductance* and denote it by g_l . The *effective resistance* between a pair of nodes i and j , denoted R_{ij} , is the electrical resistance measured across nodes i and j when the network represents an electrical circuit with each edge (or branch, in the terminology of electrical circuits) a resistor with (electrical) conductance g_l . In other words, R_{ij} is the potential difference that appears across terminals i and j when a unit current source is applied between them. We will give a formal, precise definition of effective resistance later; for now we simply note that it is a measure of how “close” the nodes i and j are: R_{ij} is small when there are many paths between nodes i and j with high conductance edges, and R_{ij} is large when there are few paths, with lower conductance, between nodes i and j . Indeed, the resistance R_{ij} is sometimes referred to as the *resistance distance* between nodes i and j .

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We define the *total effective resistance*, R_{tot} , as the sum of the effective resistance between all distinct pairs of nodes,

$$(1) \quad R_{\text{tot}} = \frac{1}{2} \sum_{i,j=1}^n R_{ij} = \sum_{i<j} R_{ij}.$$

The total effective resistance is evidently a quantitative scalar measure of how well “connected” the network is, or how “large” the network is, in terms of resistance distance. The total effective resistance comes up in a number of contexts. In an electrical network, R_{tot} is related to the average power dissipation of the circuit with a random current excitation. The total effective resistance arises in Markov chains as well: R_{tot} is, up to a scale factor, the average commute time (or average hitting time) of a Markov chain on the graph, with weights given by the edge conductances g . In this context, a network with small total effective resistance corresponds to a Markov chain with small hitting or commute times between nodes, and a large total effective resistance corresponds to a Markov chain with large hitting or commute times between at least some pairs of nodes. We will see several other applications where the total effective resistance arises, including averaging networks, experiment design, and Euclidean distance embeddings.

In this paper we address the problem of allocating a fixed total conductance among the edges so as to minimize the total effective resistance of the graph. We can assume without loss of generality that the total conductance to be allocated is 1, so we have the optimization problem

$$(2) \quad \begin{array}{ll} \text{minimize} & R_{\text{tot}} \\ \text{subject to} & \mathbf{1}^T g = 1, \quad g \geq 0. \end{array}$$

Here, the optimization variable is $g \in \mathbf{R}^m$, the vector of edge conductances, and the problem data is the graph (topology) (V, E) . The symbol $\mathbf{1}$ denotes the vector with all entries 1, and the inequality symbol \geq between vectors means componentwise inequality. We refer to problem (2) as the *effective resistance minimization problem* (ERMP).

We will give several interpretations of this problem. In the context of electrical networks, the ERMP corresponds to allocating conductance to the branches of a circuit so as to achieve low resistance between the nodes; in a Markov chain context, the ERMP is the problem of selecting the weights on the edges to minimize the average commute (or hitting) time between nodes. When R_{ij} are interpreted as distances, the ERMP is the problem of allocating conductance to a graph to make the graph “small,” in the sense of average distance between nodes.

We denote the optimal solution of the ERMP (which we will show always exists and is unique) as g^* and the corresponding optimal total effective resistance as R_{tot}^* . In [15], Fiedler introduced the general idea of optimizing some function, say, Φ , of a weighted graph over all nonnegative edge weights that add to m , i.e., with average edge weight 1. He refers to the optimal value of this problem as the *absolute Φ* of the graph. For example, if Φ is the second smallest eigenvalue of the associated Laplacian, which is called the *algebraic connectivity* of a (weighted) graph, then the *absolute algebraic connectivity* (of a graph) is the maximum value of the second eigenvalue, optimized over all weights on the edges that sum to m . Our ERMP is thus a special case of Fiedler’s construction: $(1/m)R_{\text{tot}}^*$ is what Fiedler would call the *absolute total effective resistance* of the graph.

In this paper, we will show that problem (2) is a *convex optimization problem*, which can be formulated as a semidefinite program (SDP) [7]. This has several implications, both practical and theoretical. One practical consequence is that we can solve the ERMP efficiently. On the theoretical side, convexity of the ERMP allows us to form necessary and sufficient optimality conditions and several associated dual problems (with zero duality gap). Feasible points in these dual problems give us lower bounds on R_{tot}^* ; in fact, we obtain a lower bound on R_{tot}^* given any feasible allocation of conductances. This gives us an easily computable upper bound on the suboptimality, i.e., a duality gap, given a conductance allocation g . We use this duality gap in a simple interior-point algorithm for solving the ERMP.

We describe several families of graphs for which the solution to (2) can be found analytically, by exploiting symmetry or other structure. These include trees, wheels, and the barbell graph $K_n - K_n$. For the barbell graph, we show that the ratio of R_{tot}^* to R_{tot} obtained with uniform edge weights converges to zero as the size of the graph increases. Thus, the total effective resistance of a graph, with optimized edge weights, can be unboundedly better (i.e., smaller) than the total effective resistance of a graph with uniform allocation of weights to the edges.

This paper is organized as follows. In section 2, we give a formal definition of the effective resistance, derive a number of formulas and expressions for R_{ij} , R_{tot} , and the first and second derivatives of R_{tot} , and establish several important properties, such as convexity of R_{ij} and R_{tot} as a function of the edge conductances. In section 3, we give several interpretations of R_{ij} , R_{tot} , and the ERMP.

We study the ERMP in section 4, giving the SDP formulation, (necessary and sufficient) optimality conditions, two dual problems, and a simple but effective custom interior-point method for solving it. In section 5, we study some families of graphs for which the ERMP can be solved analytically. In section 6, we show that of all graphs on n nodes, the optimal value of the ERMP is smallest for the complete graph and largest for the path. We give some numerical examples in section 7 and describe some extensions in section 8.

1.1. Related Problems. The ERMP is related to several other convex optimization problems that involve choice of some weights on the edges of a graph. One such problem (already mentioned above) is to assign nonnegative weights, which add to 1, to the edges of a graph so as to maximize the second smallest eigenvalue of the Laplacian:

$$(3) \quad \begin{aligned} & \text{maximize} && \lambda_2(L) \\ & \text{subject to} && \mathbf{1}^T g = 1, \quad g \geq 0. \end{aligned}$$

Here L denotes the Laplacian of the weighted graph. This problem has been studied in different contexts. The eigenvalue $\lambda_2(L)$ is related to the *mixing rate* of the Markov process with edge transition rates given by the edge weights. In [28], the weights g are optimized to obtain the fastest mixing Markov process on the given graph. Problem (3) has also been studied in the context of algebraic connectivity [14]. The algebraic connectivity is the second smallest eigenvalue of the Laplacian matrix L of a graph (with unit edge weights) and is a measure of how well connected the graph is. Fiedler defined the absolute algebraic connectivity of a graph as the maximum value of $\lambda_2(L)$ over all nonnegative edge weights that add up to m , i.e., $1/m$ times the optimal value of (3). The problem of finding the absolute algebraic connectivity of a graph was discussed in [15, 16], and an analytical solution was presented for tree graphs.

Other convex problems involving edge weights on graphs include the problem of finding the fastest mixing Markov chain on a given graph [6, 19, 28], the problem of finding the edge weights (which can be negative) that give the fastest convergence in an averaging network [30], and the problem of finding edge weights that give the smallest least mean-square (LMS) consensus error [31]. Convex optimization can also be used to obtain bounds on various quantities over a family of graphs; see [18].

In [8], Boyd et al. considered the sizing of the wires in the power supply network of an integrated circuit, with unknown load currents modeled stochastically. This turns out to be closely related to our ERMP, with the wire segment widths proportional to the edge weights.

Some papers on various aspects of resistance distance include [22, 32, 2, 23, 24].

2. The Effective Resistance.

2.1. Definition. Suppose edge l connects nodes i and j . We define $a_l \in \mathbf{R}^n$ as $(a_l)_i = 1$, $(a_l)_j = -1$, and all other entries 0. The *conductance matrix* (or *weighted Laplacian*) of the network is defined as

$$G = \sum_{l=1}^m g_l a_l a_l^T = A \operatorname{diag}(g) A^T,$$

where $\operatorname{diag}(g) \in \mathbf{R}^{m \times m}$ is the diagonal matrix formed from g and $A \in \mathbf{R}^{n \times m}$ is the *incidence* matrix of the graph,

$$A = [a_1 \cdots a_m].$$

Since $g_l \geq 0$, G is positive semidefinite, which we write as $G \succeq 0$. (The symbol \succeq denotes matrix inequality between symmetric matrices: $P \succeq Q$ means that $P - Q$ is positive semidefinite.) The matrix G satisfies $G\mathbf{1} = 0$, since $a_l^T \mathbf{1} = 0$ for each edge l . Thus, G has smallest eigenvalue 0, corresponding to the eigenvector $\mathbf{1}$.

Throughout this paper we make the following assumption about the edge weights:

(4) *The subgraph of edges with positive edge weights is connected.*

(If this is not the case, the effective resistance between any pair of nodes not connected by a path of edges with positive conductance is infinite, and many of our formulas are no longer valid.)

With this assumption, all other eigenvalues of G are positive. We denote the eigenvalues of G as

$$0 < \lambda_2 \leq \cdots \leq \lambda_n.$$

The nullspace of G is one-dimensional, the line along $\mathbf{1}$; its range has codimension 1, and is given by $\mathbf{1}^\perp$ (i.e., all vectors v with $\mathbf{1}^T v = 0$).

Let $G^{(k)}$ be the submatrix obtained by deleting the k th row and column of G . Our assumption (4) implies that each $G^{(k)}$ is nonsingular (see, e.g., [10]). We will refer to $G^{(k)}$ as the *reduced conductance matrix* (obtained by *grounding* node k).

Now we can define the effective resistance R_{ij} between a pair of nodes i and j . Let v be a solution of the equation

$$Gv = e_i - e_j,$$

where e_i denotes the i th unit vector, with 1 in the i th position and 0 elsewhere. This equation has a solution since $e_i - e_j$ is in the range of G . We define R_{ij} as

$$R_{ij} = v_i - v_j.$$

This is well defined; *all* solutions of $Gv = e_i - e_j$ give the same value of $v_i - v_j$. (This follows since the difference of any two solutions has the form $\alpha \mathbf{1}$ for some $\alpha \in \mathbf{R}$.) We define the *effective resistance matrix* $R \in \mathbf{R}^{n \times n}$ as the matrix with i, j entry R_{ij} . The effective resistance matrix is evidently symmetric and has diagonal entries zero, since $R_{ii} = 0$.

2.2. Effective Resistance in an Electrical Network. The term *effective resistance* (as well as several other terms used here) comes from electrical network analysis. We consider an electrical network with conductance g_l on branch (or edge) l . Let $v \in \mathbf{R}^n$ denote the vector of node potentials, and suppose a current J_i is injected into node i . The sum of the currents injected into the network must be zero in order for Kirchoff's current law to hold, i.e., we must have $\mathbf{1}^T J = 0$. The injected currents and node potentials are related by $Gv = J$. There are many solutions of this equation, but all differ by a constant vector. Thus, the potential difference between a pair of nodes is always well defined.

One way to fix the node potentials is to assign a potential zero to some node, say, the k th node. This corresponds to grounding the k th node. When this is done, the circuit equations are given by $G^{(k)}v^{(k)} = J^{(k)}$, where $G^{(k)}$ is the reduced conductance matrix, $v^{(k)}$ is the reduced potential vector, obtained by deleting the k th entry of v (which is zero), and $J^{(k)}$ is the reduced current vector, obtained by deleting the k th entry of J . In this formulation, $J^{(k)}$ has no restrictions; alternatively, we can say that J_k is implicitly defined as $J_k = -\mathbf{1}^T J^{(k)}$. From our assumption (4), $G^{(k)}$ is nonsingular, so there is a unique reduced potential vector $v^{(k)}$ for any vector of injected currents $J^{(k)}$.

Now consider the specific case when the external current is $J = e_i - e_j$, which corresponds to a one ampere current source connected from node j to node i . Any solution v of $Gv = e_i - e_j$ is a valid vector of node potentials; all of these differ by a constant. The difference $v_i - v_j$ is the same for all valid node potentials and is the voltage developed across terminals i and j . This voltage is R_{ij} , the effective resistance between nodes i and j . (The effective resistance between two nodes of a circuit is defined as the ratio of voltage across the nodes to the current flow injected into them.)

The effective resistance R_{ij} is the total power dissipated in the resistor network when $J = e_i - e_j$, i.e., a one ampere current source is applied between nodes i and j . This can be shown directly or by a power conservation argument. The voltage developed across nodes i and j is R_{ij} (by definition), so the power supplied by the current source, which is current times voltage, is R_{ij} . The power supplied by the external current source must equal the total power dissipated in the resistors of the network, so the latter is also R_{ij} .

2.3. Effective Resistance for a Tree. We first consider the case when \mathcal{N} is a tree. In this case the effective resistance between nodes i and j can be expressed as

$$R_{ij} = \sum \frac{1}{g_l},$$

where the sum is over edges that lie on the (unique) path between i and j . Therefore, the total effective resistance for a tree is given by

$$(5) \quad R_{\text{tot}} = \sum_{l=1}^m \frac{b_l}{g_l},$$

where b_l is the number of paths that contain edge l . We can express b_l in terms of the number of nodes on either side of edge l as $b_l = n_l(n - n_l)$, where n_l is the number of nodes on one side of edge l , including its endpoint. (The number of nodes on the other side is $n - n_l$.)

2.4. Some Formulas for Effective Resistance. In this section we derive several formulas for the effective resistance between a pair of nodes in a general graph as well as for the total effective resistance. Our first expressions involve the reduced conductance matrix, which we write here as \tilde{G} (since the particular node that is grounded will not matter). We form the reduced conductance matrix \tilde{G} by removing, say, the k th row and column of G . Let \tilde{v} , \tilde{e}_i , and \tilde{e}_j be, respectively, the vectors v , e_i , and e_j , each with the k th component removed. If $Gv = e_i - e_j$, then we have $\tilde{G}\tilde{v} = \tilde{e}_i - \tilde{e}_j$. This equation has a unique solution, $\tilde{v} = \tilde{G}^{-1}(\tilde{e}_i - \tilde{e}_j)$. The effective resistance between nodes i and j is given by $v_i - v_j = \tilde{v}_i - \tilde{v}_j$, i.e.,

$$(6) \quad R_{ij} = (\tilde{e}_i - \tilde{e}_j)^T \tilde{G}^{-1} (\tilde{e}_i - \tilde{e}_j).$$

(This is independent of the choice of node grounded, i.e., which row and column are removed.) When neither i nor j is k , the node that is grounded, we can write (6) as

$$R_{ij} = (\tilde{G}^{-1})_{ii} + (\tilde{G}^{-1})_{jj} - 2(\tilde{G}^{-1})_{ij}.$$

If j is k , the node that is grounded, then $\tilde{e}_j = 0$, so (6) becomes

$$R_{ij} = (\tilde{G}^{-1})_{ii}.$$

We can also write the effective resistance R_{ij} in terms of the pseudoinverse G^\dagger of G . We have

$$G^\dagger G = I - \mathbf{1}\mathbf{1}^T/n,$$

which is the projection matrix onto the range of G . (Here we use the simpler notation $\mathbf{1}\mathbf{1}^T/n$ to mean $(1/n)\mathbf{1}\mathbf{1}^T$.) Using this it can be verified that

$$(7) \quad G^\dagger = (G + \mathbf{1}\mathbf{1}^T/n)^{-1} - \mathbf{1}\mathbf{1}^T/n.$$

The following formula gives R_{ij} in terms of G^\dagger (see, e.g., [23]):

$$(8) \quad R_{ij} = (e_i - e_j)^T G^\dagger (e_i - e_j) = (G^\dagger)_{ii} + (G^\dagger)_{jj} - 2(G^\dagger)_{ij}.$$

To see this, multiply $Gv = e_i - e_j$ on the left by G^\dagger to get $(I - \mathbf{1}\mathbf{1}^T/n)v = G^\dagger(e_i - e_j)$, so

$$(e_i - e_j)^T G^\dagger (e_i - e_j) = (e_i - e_j)^T v = v_i - v_j$$

(since $e_i - e_j \perp \mathbf{1}$). From (7), we get another formula for the effective resistance,

$$(9) \quad R_{ij} = (e_i - e_j)^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} (e_i - e_j).$$

We can derive several formulas for the effective resistance matrix R , using (6) and (8). From (8), we see that

$$(10) \quad R = \mathbf{1} \operatorname{diag}(G^\dagger)^T + \operatorname{diag}(G^\dagger)\mathbf{1}^T - 2G^\dagger,$$

where $\operatorname{diag}(G^\dagger) \in \mathbf{R}^n$ is the vector consisting of the diagonal entries of G^\dagger .

Using (7), this can be rewritten as

$$(11) \quad R = \mathbf{1} \operatorname{diag}((G + \mathbf{1}\mathbf{1}^T/n)^{-1})^T + \operatorname{diag}((G + \mathbf{1}\mathbf{1}^T/n)^{-1})\mathbf{1}^T - 2(G + \mathbf{1}\mathbf{1}^T/n)^{-1}.$$

We can also derive a matrix expression for R in terms of the reduced conductance matrix \tilde{G} . Suppose \tilde{G} is formed by removing the k th row and column from G . Form a matrix $H \in \mathbf{R}^{n \times n}$ from \tilde{G}^{-1} by adding a k th row and column with all entries zero. Then, using (6), R can be written as

$$(12) \quad R = \mathbf{1} \operatorname{diag}(H)^T + \operatorname{diag}(H)\mathbf{1}^T - 2H.$$

2.5. Some Formulas for Total Effective Resistance. In this section we give several general formulas for the total effective resistance,

$$R_{\text{tot}} = \sum_{i < j} R_{ij} = (1/2)\mathbf{1}^T R \mathbf{1}.$$

From (10) we get

$$(13) \quad R_{\text{tot}} = (1/2)\mathbf{1}^T \mathbf{1} \operatorname{diag}(G^\dagger)^T \mathbf{1} + (1/2)\mathbf{1}^T \operatorname{diag}(G^\dagger)\mathbf{1}^T \mathbf{1} - \mathbf{1}^T G^\dagger \mathbf{1}$$

$$(14) \quad = n \operatorname{Tr} G^\dagger$$

$$= n \operatorname{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - n,$$

using $G^\dagger \mathbf{1} = 0$ to get the second equality and (7) to get the third equality. ($\operatorname{Tr} Z$ denotes the trace of a square matrix Z .)

We can use (13) to get a formula for R_{tot} in terms of the eigenvalues of G . The eigenvalues of G^\dagger are $1/\lambda_i$ for $i = 2, \dots, n$ and 0. So we can rewrite (13) as

$$(15) \quad R_{\text{tot}} = n \sum_{i=2}^n \frac{1}{\lambda_i}.$$

This expression for the total effective resistance can be found in [1, section 3.4].

The total effective resistance can also be expressed in terms of the reduced conductance matrix \tilde{G} . Multiplying (12) on the left and right by $\mathbf{1}^T$ and $\mathbf{1}$ and dividing by 2, we have

$$(16) \quad R_{\text{tot}} = n \operatorname{Tr} \tilde{G}^{-1} - \mathbf{1}^T \tilde{G}^{-1} \mathbf{1} = n \operatorname{Tr}(I - \mathbf{1}\mathbf{1}^T/n)\tilde{G}^{-1}.$$

(Note that $\tilde{G} \in \mathbf{R}^{(n-1) \times (n-1)}$, so the vectors denoted $\mathbf{1}$ in this formula have dimension $n - 1$.)

The total effective resistance can also be written in terms of an integral:

$$(17) \quad R_{\text{tot}} = n \operatorname{Tr} \int_0^\infty (e^{-tG} - \mathbf{1}\mathbf{1}^T/n) dt.$$

This can be seen as follows. Let $n^{-1/2}\mathbf{1}, v_2, \dots, v_n$ be an orthonormal set of eigenvectors of G , corresponding to the eigenvalues $\lambda_1 = 0 < \lambda_2 \leq \dots \leq \lambda_n$, respectively. The matrix e^{-tG} has the same eigenvectors, with corresponding eigenvalues 1 and $e^{-\lambda_i t}$ for $i = 2, \dots, n$. Therefore we have

$$\begin{aligned} n \operatorname{Tr} \int_0^\infty (e^{-tG} - \mathbf{1}\mathbf{1}^T/n) dt &= n \operatorname{Tr} \int_0^\infty \sum_{i=2}^n e^{-\lambda_i t} v_i v_i^T dt \\ &= n \sum_{i=2}^n \int_0^\infty e^{-\lambda_i t} dt = n \sum_{i=2}^n \frac{1}{\lambda_i}, \end{aligned}$$

using $\operatorname{Tr} v_i v_i^T = \|v_i\|^2 = 1$ to get the second equality.

2.6. Basic Properties. The effective resistance R_{ij} and the total effective resistance R_{tot} are rational functions of g . This can be seen from (6), since the inverse of a matrix is a rational function of the matrix and R_{ij} is a linear function of \tilde{G}^{-1} . They are also homogeneous with degree -1 : if $\hat{g} = cg$, where $c > 0$, then $\hat{R}_{ij} = R_{ij}/c$ and $\hat{R}_{\text{tot}} = R_{\text{tot}}/c$.

The effective resistance R_{ij} with $i \neq j$ is always positive: the matrix \tilde{G}^{-1} is positive definite (since $\tilde{G} \succ 0$), so from (6), $R_{ij} > 0$ when $i \neq j$. Nonnegativity of R_{ij} can also be seen by noting that \tilde{G} is an M -matrix. The inverse of an M -matrix is elementwise nonnegative [21], and since R_{ij} is the (i, i) th entry of $(\tilde{G}^{(j)})^{-1}$, it is nonnegative as well.

The effective resistance also satisfies the triangle inequality (see, e.g., [23]),

$$(18) \quad R_{ik} \leq R_{ij} + R_{jk}.$$

Therefore, the effective resistance defines a metric on the graph, called the *resistance distance* [23]. It also satisfies the weaker triangle inequality, $R_{ik}^{1/2} \leq R_{ij}^{1/2} + R_{jk}^{1/2}$, which arises in Euclidean distance problems. (This is discussed in section 2.8.)

The effective resistance R_{ij} is a monotone decreasing function of g , i.e., if $g \leq \hat{g}$, then $R_{ij} \geq \hat{R}_{ij}$. To show this, suppose $0 \leq g \leq \hat{g}$ and let G and \hat{G} denote the associated conductance matrices. Evidently we have $G + \mathbf{1}\mathbf{1}^T/n \preceq \hat{G} + \mathbf{1}\mathbf{1}^T/n$, so $(G + \mathbf{1}\mathbf{1}^T/n)^{-1} \succeq (\hat{G} + \mathbf{1}\mathbf{1}^T/n)^{-1}$. From (9),

$$R_{ij} = (e_i - e_j)^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} (e_i - e_j) \geq (e_i - e_j)^T (\hat{G} + \mathbf{1}\mathbf{1}^T/n)^{-1} (e_i - e_j) = \hat{R}_{ij}.$$

2.7. Convexity. The effective resistance R_{ij} is a convex function of g : for $g, \hat{g} \geq 0$ (both satisfying the basic assumption (4)) and any $\theta \in [0, 1]$, we have

$$R_{ij}(\theta g + (1 - \theta)\hat{g}) \leq \theta R_{ij}(g) + (1 - \theta)R_{ij}(\hat{g}).$$

To show this, we first observe that $f(Y) = c^T Y^{-1} c$, where $Y = Y^T \in \mathbf{R}^{n \times n}$ and $c \in \mathbf{R}^n$, is a convex function of Y for $Y \succ 0$ (see, e.g., [7, section 3.1.7]). Since $G + \mathbf{1}\mathbf{1}^T/n$ is an affine function of g , R_{ij} is a convex function of g . It follows that R_{tot} is also convex, since it is a sum of convex functions.

A related result was shown by Shannon and Hagelberger in [26]: if $r_l = 1/g_l$ is the value of the resistor on edge l , then R_{ij} is a concave function of the vector r . Note that the convexity of R_{ij} as a function of g does not follow immediately (for example, using simple composition rules) from this result.

The total effective resistance is, in fact, a *strictly* convex function of g : for $g, \hat{g} \geq 0$ (both satisfying the basic assumption (4)), with $g \neq \hat{g}$, and any $\theta \in (0, 1)$, we have

$$R_{\text{tot}}(\theta g + (1 - \theta)\hat{g}) < \theta R_{\text{tot}}(g) + (1 - \theta)R_{\text{tot}}(\hat{g}).$$

To establish this, we first show that $\mathbf{Tr} X^{-1}$ is a strictly convex function of X for X symmetric and positive definite. Its second order Taylor approximation is

$$\mathbf{Tr}(X + \Delta)^{-1} \approx \mathbf{Tr} X^{-1} - \mathbf{Tr} X^{-1} \Delta X^{-1} + \mathbf{Tr} X^{-1} \Delta X^{-1} \Delta X^{-1},$$

where $\Delta = \Delta^T$. The second order term can be expressed as

$$\mathbf{Tr} X^{-1} \Delta X^{-1} \Delta X^{-1} = \|X^{-1} \Delta X^{-1/2}\|_F^2,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. This second order term vanishes only if $\Delta = 0$ (since X^{-1} and $X^{-1/2}$ are both invertible), i.e., it is a positive definite quadratic

function of Δ . This shows that $\mathbf{Tr} X^{-1}$ is a strictly convex function of $X = X^T \succ 0$. Since the affine mapping from g to $G + \mathbf{1}\mathbf{1}^T/n$ is one-to-one, we conclude that

$$R_{\text{tot}} = n \mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - n$$

is a strictly convex function of g .

We note that the effective resistance R_{ij} need not be a strictly convex function of g . As a simple example, consider a ring with 4 nodes and consider the effective resistance between two nonadjacent nodes. There are only two paths between these nodes, one passing through edges with conductance g_1 and g_2 (say), and the other path passing through edges with conductance g_3 and g_4 . This resistance between the nodes can be expressed as

$$R = \frac{(1/g_1 + 1/g_2)(1/g_3 + 1/g_4)}{1/g_1 + 1/g_2 + 1/g_3 + 1/g_4}.$$

(This follows from direct calculation or from the formulas for the resistance of series and parallel connections of resistors.) Now consider the two edge conductance vectors $g = (1, 1, 3, 3)$ and $\hat{g} = (3, 3, 1, 1)$. Each gives resistance $R = 1/2$. With edge conductance $(g + \hat{g})/2 = (2, 2, 2, 2)$, we also have resistance $R = 1/2$, which shows that R is not strictly convex.

The convexity of R_{tot} can also be seen from the theory of *spectral functions* (i.e., a symmetric function of the eigenvalues of a symmetric matrix). From (15), we see that R_{tot} is a spectral function of G associated with the closed convex function

$$f(y) = \begin{cases} \sum_{i=2}^n 1/[y]_i, & [y]_1 = 0, \quad [y]_i > 0, \quad i = 2, \dots, n, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $[y]_i$ denotes the i th smallest entry of the vector y . A spectral function $f(\lambda)$ is closed and convex if and only if f is closed and convex [4], so R_{tot} is a convex function of G and therefore also of g .

2.8. Euclidean Distance Matrix. A symmetric matrix $D \in \mathbf{R}^{n \times n}$ is a *Euclidean distance matrix* if there is a set of n vectors $x_1, \dots, x_n \in \mathbf{R}^n$ such that $D_{ij} = \|x_i - x_j\|^2$. A classical result is that a matrix D is a Euclidean distance matrix if and only if it has nonnegative elements, zero diagonal elements, and is negative semidefinite on $\mathbf{1}^\perp$, i.e., $x^T D x \leq 0$ if $\mathbf{1}^T x = 0$ [20].

The effective resistance matrix R is a Euclidean distance matrix: it has zero diagonal, nonnegative entries and, from (10), for any x satisfying $\mathbf{1}^T x = 0$,

$$x^T R x = x^T (\mathbf{1} \text{diag}(G^\dagger)^T + \text{diag}(G^\dagger) \mathbf{1}^T - 2G^\dagger) x = -2x^T G^\dagger x \leq 0$$

since $G^\dagger \succeq 0$. In fact we can easily construct a set of vectors x_1, \dots, x_n for which

$$(19) \quad \|x_i - x_j\|^2 = R_{ij}.$$

Let x_i be the columns of $(G^\dagger)^{1/2}$, i.e.,

$$[x_1 \cdots x_n] = (G^\dagger)^{1/2}.$$

Then the Gram matrix associated with these vectors is G^\dagger :

$$x_i^T x_j = (G^\dagger)_{ij}.$$

Expanding the left-hand side of (19) gives

$$\|x_i - x_j\|^2 = x_i^T x_i + x_j^T x_j - 2x_i^T x_j = (G^\dagger)_{ii} + (G^\dagger)_{jj} - 2(G^\dagger)_{ij},$$

which is R_{ij} , by (8). (See [7, section 8.3.3].) A relation between R and Euclidean distance matrices is also discussed in [17].

Since R is a Euclidean distance matrix, the square roots of the entries of R are Euclidean distances, i.e., they satisfy the triangle inequality

$$R_{ik}^{1/2} \leq R_{ij}^{1/2} + R_{jk}^{1/2}$$

(which also follows directly from the stronger triangle inequality (18)). We refer to $R_{ij}^{1/2}$ as the *Euclidean distance* between nodes i and j . Thus, R_{tot} is the sum of the squares of the Euclidean distances between all distinct pairs of nodes.

2.9. Gradient and Hessian. In this section we work out some formulas for the gradient and Hessian of R_{tot} with respect to g . (A similar approach can be used to find the derivatives of R_{ij} with respect to g , but we will not need these in what follows.) We will use the following fact. Suppose the invertible symmetric matrix $X(t)$ is a differentiable function of the parameter $t \in \mathbf{R}$. Then we have [7, section A.4.1]

$$\frac{\partial X^{-1}}{\partial t} = -X^{-1} \frac{\partial X}{\partial t} X^{-1}.$$

Using this formula and $R_{\text{tot}} = n \mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - n$, we have

$$\begin{aligned} \frac{\partial R_{\text{tot}}}{\partial g_l} &= -n \mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} \frac{\partial G}{\partial g_l} (G + \mathbf{1}\mathbf{1}^T/n)^{-1} \\ &= -n \mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} \\ (20) \quad &= -n \|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2. \end{aligned}$$

We can express the gradient as

$$\nabla R_{\text{tot}} = -n \text{diag}(A^T (G + \mathbf{1}\mathbf{1}^T/n)^{-2} A).$$

The gradient can also be expressed in terms of a reduced conductance matrix. Let $\tilde{G} = G^{(k)}$ be a reduced conductance matrix with node k grounded, and let $\tilde{a}_l \in \mathbf{R}^{n-1}$ be the corresponding reduced edge vectors and \tilde{A} the corresponding graph incidence matrix. Using $R_{\text{tot}} = n \mathbf{Tr} \tilde{G}^{-1} (I - \mathbf{1}\mathbf{1}^T/n)$, we have

$$\frac{\partial R_{\text{tot}}}{\partial g_l} = -n \mathbf{Tr}(I - \mathbf{1}\mathbf{1}^T/n) \tilde{G}^{-1} \frac{\partial \tilde{G}}{\partial g_l} \tilde{G}^{-1} = -n \mathbf{Tr}(I - \mathbf{1}\mathbf{1}^T/n) \tilde{G}^{-1} \tilde{a}_l \tilde{a}_l^T \tilde{G}^{-1}$$

(here $I - \mathbf{1}\mathbf{1}^T/n \in \mathbf{R}^{(n-1) \times (n-1)}$). Thus, the gradient can be written as

$$\nabla R_{\text{tot}} = -n \text{diag}(\tilde{A}^T \tilde{G}^{-1} (I - \mathbf{1}\mathbf{1}^T/n) \tilde{G}^{-1} \tilde{A}).$$

For future reference, we note the formula

$$(21) \quad \nabla R_{\text{tot}}^T g = -R_{\text{tot}},$$

which holds since R_{tot} is a homogeneous function of g of degree -1 . It is easily verified by taking the derivative with respect to α of $R_{\text{tot}}(\alpha g) = R_{\text{tot}}(g)/\alpha$, evaluated at $\alpha = 1$.

We now derive the second derivative or Hessian matrix of R_{tot} . From (20), we have

$$(22) \quad \frac{\partial^2}{\partial g_l \partial g_k} R_{\text{tot}} = -n \frac{\partial}{\partial g_k} \|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2$$

$$(23) \quad = 2na_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-2} a_k a_k^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l.$$

We can express the Hessian of R_{tot} as

$$\nabla^2 R_{\text{tot}} = 2n (A^T (G + \mathbf{1}\mathbf{1}^T/n)^{-2} A) \circ (A^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} A),$$

where \circ denotes the Hadamard (elementwise) product. A similar expression can be derived using reduced matrices:

$$(24) \quad \nabla^2 R_{\text{tot}} = 2n (\tilde{A}^T \tilde{G}^{-1} (I - \mathbf{1}\mathbf{1}^T/n) \tilde{G}^{-1} \tilde{A}) \circ (\tilde{A}^T \tilde{G}^{-1} \tilde{A}).$$

Suppose edge l connects nodes i and j . From the formulas above, and using $R_{ij} = a_i^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_j$, we have

$$\frac{\partial^2 R_{\text{tot}}}{\partial^2 g_l} = -2R_{ij} \frac{\partial R_{\text{tot}}}{\partial g_l}.$$

3. Interpretations.

3.1. Average Commute Time. The effective resistance between a pair of nodes i and j is related to the commute time between i and j for the Markov chain defined by the conductances g [9, 11]. Let \mathcal{M} be a Markov chain on the graph \mathcal{N} with transition probabilities determined by the conductances,

$$P_{ij} = \frac{g_{ij}}{\sum_{l \sim (i,k)} g_l},$$

where g_{ij} is the conductance across edge (i, j) and $l \sim (i, k)$ means that edge l lies between nodes i and k . This Markov chain is reversible, with stationary distribution

$$\pi_i = \frac{\sum_{l \sim (i,j)} g_l}{\sum_{l=1}^m g_l}.$$

The *hitting time* H_{ij} is the (random) time taken to reach node j for the first time starting from node i . The *commute time* C_{ij} is the time it takes to return to node i for the first time after starting from i and passing through node j . The following well-known result relates commute times and effective resistance (see, for example, [1, section 3.3]):

$$\mathbf{E} C_{ij} = 2(\mathbf{1}^T g) R_{ij},$$

where \mathbf{E} denotes expected value. That is, the effective resistance between i and j is proportional to the expected commute time between i and j .

Therefore, the total effective resistance is proportional to \bar{C} , the expected commute time averaged over all pairs of nodes:

$$\bar{C} = \frac{4(\mathbf{1}^T g)}{n(n-1)} R_{\text{tot}}.$$

Since $\mathbf{E} C_{ij} = \mathbf{E} H_{ij} + \mathbf{E} H_{ji}$, R_{tot} is also proportional to \overline{H} , the expected hitting time averaged over all pairs of nodes:

$$\overline{H} = \frac{2(\mathbf{1}^T g)}{n(n-1)} R_{\text{tot}}.$$

In the context of Markov chains, the ERMP (2) is the problem of choosing edge weights on a graph so as to minimize its average expected commute or hitting time.

3.2. Power Dissipation in a Resistor Network. The total effective resistance is related to the average power dissipated in a resistor network with random injected currents. Suppose a random current $J \in \mathbf{R}^n$ is injected into the network. The current must satisfy $\mathbf{1}^T J = 0$, since the total current entering the network must be zero. We assume that

$$\mathbf{E} J = 0, \quad \mathbf{E} J J^T = I - \mathbf{1}\mathbf{1}^T/n.$$

Roughly speaking, this means J is a random current vector, with covariance matrix I on $\mathbf{1}^\perp$.

The power dissipated in the resistor network with injected current vector J is $J^T G^\dagger J$. The expected dissipated power is

$$\mathbf{E} J^T G^\dagger J = \mathbf{Tr} G^\dagger \mathbf{E} J J^T = \mathbf{Tr} G^\dagger = \frac{1}{n} R_{\text{tot}},$$

where the second equality follows from $G^\dagger \mathbf{1} = 0$.

Thus, the total effective resistance is proportional to the average power dissipated in the network when the injected current is random, with mean 0 and covariance $I - \mathbf{1}\mathbf{1}^T/n$. A network with small R_{tot} is one which dissipates little power under random current excitation; large R_{tot} means the average power dissipation is large.

The ERMP (2) is the problem of allocating a total conductance of 1 to the branches of a resistor network so as to minimize the average power dissipated under random current excitation. (See, e.g., [8].)

We can also give an interpretation of the gradient ∇R_{tot} in the context of a resistor network. With random current excitation J , with $\mathbf{E} J = 0$, $\mathbf{E} J J^T = I - \mathbf{1}\mathbf{1}^T/n$, the partial derivative $\partial R_{\text{tot}}/\partial g_l$ is proportional to the mean-square voltage across edge l . This can be seen from (20) as follows. The voltage v_l across edge l with current excitation J is $a_l^T G^\dagger J = a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} J$, since $a_l^T \mathbf{1} = 0$. The expected value of the squared voltage is

$$\begin{aligned} \mathbf{E}(a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} J)^2 &= a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} \mathbf{E} J J^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l \\ &= a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} (I - \mathbf{1}\mathbf{1}^T/n) (G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l \\ &= a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-2} a_l, \end{aligned}$$

where the last equality follows since $(G + \mathbf{1}\mathbf{1}^T/n)^{-1} \mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T a_l = 0$. Comparing this with (20), we see that

$$(25) \quad \frac{\partial R_{\text{tot}}}{\partial g_l} = -n \mathbf{E} v_l^2.$$

The gradient ∇R_{tot} is equal to $-n$ times the vector of mean-square voltages appearing across the edges.

3.3. Elmore Delay in an RC Circuit. We consider again a resistor network with branch (electrical) conductances given by g_i . To this network we add a separate ground node and a unit capacitance between every other node and the ground node. The vector of node voltages (with respect to the ground node) in this resistor-capacitor (RC) circuit evolves according to $\dot{v} = -Gv$. This has solution $v(t) = e^{-tG}v(0)$. Since e^{-tG} has largest eigenvalue 1 associated with the eigenvector $\mathbf{1}$, with other eigenvalues $e^{-\lambda_i t}$ for $i = 2, \dots, n$, we see that $v(t)$ converges to the vector $\mathbf{1}\mathbf{1}^T v(0)/n$. In other words, the voltage (or, equivalently, charge) equilibrates itself across the nodes in the circuit.

Suppose we start with the initial voltage $v(0) = e_k$, i.e., one volt on node k with zero voltage on all other nodes. It can be shown that the voltage at node k monotonically decreases to the average value, $1/n$. The *Elmore delay* at node k is defined as

$$T_k = \int_0^\infty (v_k(t) - 1/n) dt$$

(see, for example, [12, 29]). The Elmore delay T_k gives a measure of the speed at which charge starting at node k equilibrates.

The average Elmore delay, over all nodes, is

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n T_k &= \frac{1}{n} \sum_{k=1}^n e_k^T \int_0^\infty (e^{-tG} - \mathbf{1}\mathbf{1}^T/n) e_k dt \\ &= \frac{1}{n} \text{Tr} \left(\int_0^\infty (e^{-tG} - \mathbf{1}\mathbf{1}^T/n) dt \right) \\ &= \frac{1}{n^2} R_{\text{tot}}, \end{aligned}$$

where the last equality follows from (17). The total effective resistance of the network is thus proportional to the sum of the Elmore delays at each node in the RC circuit.

The ERMP (2) is the problem of allocating a total conductance of 1 to the resistor branches of an RC circuit so as to minimize the average Elmore delay of the nodes.

3.4. Total Time Constant of an Averaging Network. We can interpret R_{tot} in terms of the time constants in an averaging network. We consider the dynamical system $\dot{x} = -Gx$, where G is the conductance matrix. This system carries out (asymptotic) averaging: e^{-tG} is a doubly stochastic matrix, which converges to $\mathbf{1}\mathbf{1}^T/n$ as $t \rightarrow \infty$, so $x(t) = e^{-tG}x(0)$ converges to $\mathbf{1}\mathbf{1}^T x(0)/n$.

The eigenvalues $\lambda_2, \dots, \lambda_n$ of G determine the rate at which the averaging takes place. The eigenvectors v_2, \dots, v_n are the modes of the system, and the associated time constants are given by $T_k = 1/\lambda_k$. (This gives the time for mode k to decay by a factor e .) Therefore, $R_{\text{tot}} = n \sum_{k=2}^n T_k$ is proportional to the sum of the time constants of the averaging system.

3.5. A-Optimal Experiment Design. The ERMP can be interpreted as a certain type of optimal experiment design problem. The goal is to estimate a parameter vector $x \in \mathbf{R}^n$ from noisy linear measurements

$$y_i = v_i^T x + w_i, \quad i = 1, \dots, K,$$

where each v_i can be any of the vectors a_1, \dots, a_m and w_i are independent random (noise) variables with zero mean and unit variance. Thus, each measurement consists

of measuring a difference between two components of x , corresponding to some edge of our graph, with some additive noise. With these measurements of differences of components, we can only estimate x up to some additive constant; the parameters x and $x + \alpha \mathbf{1}$ for any $\alpha \in \mathbf{R}$ produce exactly the same measurements. We will therefore assume that the parameter to be estimated satisfies $\mathbf{1}^T x = 0$.

Now suppose a total of k_l measurements are made using a_l for $l = 1, \dots, m$, so we have $\sum_{l=1}^m k_l = K$. The minimum variance unbiased estimate of x , given the measurements, is

$$\hat{x} = \left(\sum_{i=1}^K v_i v_i^T \right)^\dagger \left(\sum_{i=1}^K v_i^T y_i \right) = \left(\sum_{l=1}^m k_l a_l a_l^T \right)^\dagger \left(\sum_{l=1}^m k_l a_l^T y \right).$$

The associated estimation error $e = \hat{x} - x$ has zero mean and covariance matrix

$$\Sigma_{\text{err}} = \left(\sum_{l=1}^m k_l a_l a_l^T \right)^\dagger.$$

(There is no estimation error in the direction $\mathbf{1}$, since we have assumed that $\mathbf{1}^T x = 0$, and we always have $\mathbf{1}^T \hat{x} = 0$.) The goal of experiment design is to choose the integers k_1, \dots, k_m , subject to $\sum_l k_l = K$, to make the estimation error covariance matrix Σ_{err} small. There are several ways to define “small,” which yield different experiment design problems. In A -optimal experiment design, the objective is the trace of Σ_{err} . This is proportional to the sum of the squares of the semiaxis lengths of the confidence ellipsoid associated with the estimate \hat{x} .

We now change variables to $\theta_l = k_l/K$, which is the fraction of the total number of experiments (i.e., K) that are carried out using $v = a_l$. The variables θ_l are nonnegative and add to 1, and they must be integer multiples of $1/K$. If K is large, we can ignore the last requirement and take the variables θ_l to be real. This yields the (relaxed) A -optimal experiment design problem [25]

$$\begin{aligned} & \text{minimize} && (1/K) \mathbf{Tr}(\sum_{l=1}^m \theta_l a_l a_l^T)^\dagger \\ & \text{subject to} && \sum_{l=1}^m \theta_l = 1, \quad \theta_l \geq 0. \end{aligned}$$

This is a convex optimization problem, with variable $\theta \in \mathbf{R}^m$. (See [7, section 7.5] and its references for more on experiment design problems.)

Identifying θ_l with g_l , we see that the A -optimal experiment design problem above is the same as our ERMP (up to scale factor in the objective). Thus, we can interpret the ERMP as follows. We have real numbers x_1, \dots, x_n at the nodes of our graph, which have zero sum. Each edge in our graph corresponds to a possible measurement we can make, which gives the difference in its adjacent node values, plus a noise. We make a large number of these measurements, in order to estimate x . The problem is to choose the fraction of the experiments that should be devoted to each edge measurement. Using the trace of the error covariance matrix as our measure of estimation quality, the optimal fractions are exactly the optimal conductances in the ERMP.

3.6. Euclidean Variance. In section 2.8 we observed that R is a Euclidean distance matrix and interpreted $R_{ij}^{1/2}$ as the Euclidean distance between nodes i and j . The total effective resistance R_{tot} is the sum of the squares of the Euclidean distances between all distinct pairs of nodes. Thus, the ERMP is the problem of choosing edge weights, which add to 1, that makes the graph small, i.e., minimizes the sum of the squares of the Euclidean distances between all distinct pairs of nodes.

We can give another, related interpretation of the objective. Since the coordinates of the n points x_1, \dots, x_n are the columns of $(G^\dagger)^{1/2}$, the coordinates satisfy $\sum_{i=1}^n x_i = 0$, i.e., they have zero mean. The total effective resistance is therefore the same as

$$R_{\text{tot}} = \sum_{i < j} \|x_i - x_j\|^2 = n \sum_{i=1}^n \|x_i\|^2,$$

which is n times the *variance* of the configuration of n points. Thus, the ERMP is to choose edge weights, which add to 1, to minimize the variance of the associated Euclidean configuration. (We note that the variance is the same for *any* configuration of points that satisfy $\|x_i - x_j\|^2 = R_{ij}$.)

4. Minimizing Total Effective Resistance. In this section we study the ERMP

$$(26) \quad \begin{array}{ll} \text{minimize} & R_{\text{tot}} \\ \text{subject to} & \mathbf{1}^T g = 1, \quad g \geq 0, \end{array}$$

in detail. This is a convex optimization problem, since the objective is a convex function of g and the constraint functions are linear. The problem is clearly feasible, since $g = (1/m)\mathbf{1}$, the uniform allocation of conductance to edges, is feasible. Since the objective function is strictly convex, the solution of (26) is unique. We denote the unique optimal point as g^* and the associated value of the objective as R_{tot}^* .

4.1. SDP Formulation. The ERMP (26) can be formulated as an SDP,

$$(27) \quad \begin{array}{ll} \text{minimize} & n \mathbf{Tr} Y \\ \text{subject to} & \mathbf{1}^T g = 1, \quad g \geq 0, \\ & \begin{bmatrix} G + \mathbf{1}\mathbf{1}^T/n & I \\ I & Y \end{bmatrix} \succeq 0, \end{array}$$

where $G = \sum_{l=1}^m g_l a_l a_l^T$. The variables are the conductances $g \in \mathbf{R}^m$ and the slack symmetric matrix $Y \in \mathbf{R}^{n \times n}$. To see the equivalence, we note that whenever $G + \mathbf{1}\mathbf{1}^T/n \succ 0$ (which is guaranteed whenever the basic assumption (4) holds),

$$\begin{bmatrix} G + \mathbf{1}\mathbf{1}^T/n & I \\ I & Y \end{bmatrix} \succeq 0 \iff Y \succeq (G + \mathbf{1}\mathbf{1}^T/n)^{-1}.$$

To minimize the SDP objective $n \mathbf{Tr} Y$, subject to this constraint, with G fixed, we simply take $Y = (G + \mathbf{1}\mathbf{1}^T/n)^{-1}$, so the objective of the SDP becomes $R_{\text{tot}} + n$.

4.2. Optimality Conditions. The optimal conductance g^* satisfies

$$(28) \quad \mathbf{1}^T g = 1, \quad g \geq 0, \quad \nabla R_{\text{tot}} + R_{\text{tot}} \mathbf{1} \geq 0,$$

where R_{tot} is the total effective resistance with g . Conversely, if g is any vector of conductances that satisfies (28), then it is optimal, i.e., $g = g^*$. The first two conditions in (28) require that g be feasible.

These optimality conditions can be derived as follows. Since the ERMP is a convex problem with differentiable objective, a necessary and sufficient condition for optimality of a feasible g is

$$\nabla R_{\text{tot}}^T (\hat{g} - g) \geq 0 \quad \text{for all } \hat{g} \text{ with } \mathbf{1}^T \hat{g} = 1, \hat{g} \geq 0$$

(see, e.g., [7, section 4.2.3]). This is the same as

$$\nabla R_{\text{tot}}^T (e_l - g) \geq 0, \quad l = 1, \dots, m.$$

Since R_{tot} is a homogeneous function of g of degree -1 , we have $\nabla R_{\text{tot}}^T g = -R_{\text{tot}}$ (see (21)), so the condition above can be written as

$$(29) \quad \frac{\partial R_{\text{tot}}}{\partial g_l} + R_{\text{tot}} \geq 0, \quad l = 1, \dots, m,$$

which is precisely the third condition in (28). This third condition can be expressed as

$$\frac{\partial R_{\text{tot}}}{\partial g_l} + R_{\text{tot}} = n \left(\text{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - 1 - \|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2 \right) \geq 0, \quad l = 1, \dots, m.$$

From the optimality conditions (28) we can derive a complementary slackness condition:

$$(30) \quad g_l \left(\frac{\partial R_{\text{tot}}}{\partial g_l} + R_{\text{tot}} \right) = 0, \quad l = 1, \dots, m.$$

This means that for each edge, we have either $g_l = 0$ or $\partial R_{\text{tot}}/\partial g_l + R_{\text{tot}} = 0$. To establish the complementarity condition, we note that

$$g^T (\nabla R_{\text{tot}} + R_{\text{tot}} \mathbf{1}) = 0,$$

since $g^T \nabla R_{\text{tot}} = -R_{\text{tot}}$ and $g^T R_{\text{tot}} \mathbf{1} = R_{\text{tot}}$. If g satisfies (28), then this states that the inner product of two nonnegative vectors, g and $\nabla R_{\text{tot}} + R_{\text{tot}} \mathbf{1}$, is zero; it follows that the products of the corresponding entries are zero. This is exactly the complementarity condition above.

We can give the optimality conditions a simple interpretation in the context of a circuit driven by a random current, as described in section 3.2. We suppose the circuit is driven by a random current excitation J with zero mean and covariance $\mathbf{E} J J^T = I - \mathbf{1}\mathbf{1}^T/n$. By (25), we have $\partial R_{\text{tot}}/\partial g_l = -n \mathbf{E} v_l^2$, where v_l is the (random) voltage appearing across edge l . The optimality condition is that g is feasible, and we have

$$\mathbf{E} v_l^2 \leq (1/n) R_{\text{tot}}, \quad l = 1, \dots, m.$$

Thus, the conductances are optimal when the mean-square voltage across each edge is less than or equal to $(1/n) R_{\text{tot}}$. Using the complementarity condition (30), we can be a bit more specific: each edge that has positive conductance allocated to it must have a mean-square voltage equal to $(1/n) R_{\text{tot}}$; any edge with zero conductance must have a mean-square voltage no more than $(1/n) R_{\text{tot}}$.

4.3. The Dual Problem. In this section we derive the Lagrange dual problem for the ERMP (26), as well as some interesting variations on it. We start by writing the ERMP as

$$(31) \quad \begin{aligned} & \text{minimize} && n \text{Tr} X^{-1} - n \\ & \text{subject to} && X = \sum_{l=1}^m g_l a_l a_l^T + \mathbf{1}\mathbf{1}^T/n, \\ & && \mathbf{1}^T g = 1, \quad g \geq 0, \end{aligned}$$

with variables $g \in \mathbf{R}^m$ and $X = X^T \in \mathbf{R}^{n \times n}$. Associating dual variables $Z = Z^T \in \mathbf{R}^{n \times n}$, $\nu \in \mathbf{R}$ with the equality constraints, and $\lambda \in \mathbf{R}^m$ with the nonnegativity constraint $g \geq 0$, the Lagrangian is

$$L(X, g, Z, \nu, \lambda) = n \operatorname{Tr} X^{-1} - n + \operatorname{Tr} Z \left(X - \sum_{l=1}^m g_l a_l a_l^T - \mathbf{1}\mathbf{1}^T/n \right) + \nu(\mathbf{1}^T g - 1) - \lambda^T g.$$

The dual function is

$$\begin{aligned} h(Z, \nu, \lambda) &= \inf_{X \succeq 0, g} L(X, g, Z, \nu, \lambda) \\ &= \inf_{X \succeq 0} \operatorname{Tr}(nX^{-1} + ZX) + \inf_g \left(\sum_{l=1}^m g_l (-a_l^T Z a_l + \nu - \lambda_l) \right) - n - \nu - (1/n) \mathbf{1}^T Z \mathbf{1} \\ &= \begin{cases} -\nu - (1/n) \mathbf{1}^T Z \mathbf{1} + 2 \operatorname{Tr}(nZ)^{1/2} - n & \text{if } -a_l^T Z a_l + \nu = \lambda_l, \quad l = 1, \dots, m, \quad Z \succeq 0, \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

To justify the last line, we note that $\operatorname{Tr}(nX^{-1} + ZX)$ is unbounded below, as a function of X , unless $Z \succeq 0$; when $Z \succ 0$, the unique X that minimizes it is $X = (Z/n)^{-1/2}$, so it has the value

$$\operatorname{Tr}(nX^{-1} + ZX) = \operatorname{Tr}(n(Z/n)^{1/2} + Z(Z/n)^{-1/2}) = 2 \operatorname{Tr}(nZ)^{1/2}.$$

When Z is positive semidefinite but not positive definite, we get the same minimal value, but it is not achieved by any X . (This calculation is equivalent to working out the conjugate of the function $\operatorname{Tr} U^{-1}$ for $U \succ 0$, which is $-2 \operatorname{Tr}(-V)^{1/2}$ with domain $V \preceq 0$; see, e.g., [7, Ex. 3.37].)

The Lagrange dual problem is

$$\begin{aligned} &\text{maximize} && h(Z, \nu, \lambda) \\ &\text{subject to} && \lambda \geq 0. \end{aligned}$$

Using the explicit formula for h derived above and eliminating λ , which serves as a slack variable, we obtain the dual problem

$$(32) \quad \begin{aligned} &\text{maximize} && -\nu - (1/n) \mathbf{1}^T Z \mathbf{1} + 2 \operatorname{Tr}(nZ)^{1/2} - n \\ &\text{subject to} && a_l^T Z a_l \leq \nu, \quad l = 1, \dots, m, \\ &&& Z \succeq 0. \end{aligned}$$

This problem is another convex optimization problem with variables $Z = Z^T \in \mathbf{R}^{n \times n}$ and $\nu \in \mathbf{R}$. The scalar variable ν could be eliminated, since its optimal value is evidently $\nu = \max_l a_l^T Z a_l$.

Since the ERMP is convex, has only linear equality and inequality constraints, and Slater's condition is satisfied (for example, by $g = (1/m)\mathbf{1}$), we know that the optimal duality gap for the ERMP (31) and the dual problem (32) is zero. In other words, the optimal value of the dual (32) is equal to R_{tot}^* , the optimal value of the ERMP. In fact, we can be very explicit: if X^* is the optimal solution of the primal ERMP (31), then

$$Z^* = n(X^*)^{-2}, \quad \nu^* = \max_l a_l^T Z^* a_l$$

are optimal for the dual ERMP (32). Conversely, if Z^* is optimal for the dual ERMP (32), then $X^* = (Z^*/n)^{-1/2}$ is the optimal point for the primal ERMP (31).

We can use the dual problem (32) to derive a useful bound on the suboptimality of any feasible conductance vector g by constructing a dual feasible point from g . With $G = A \operatorname{diag}(g) A^T$, we define

$$Z = n(G + \mathbf{1}\mathbf{1}^T/n)^{-2}$$

with $\nu = \max_l a_l^T Z a_l$. The pair (Z, ν) is evidently feasible for the dual problem, so its dual objective value gives a lower bound \underline{R} on R_{tot}^* :

$$\begin{aligned} R_{\text{tot}}^* &\geq -\nu - (1/n)\mathbf{1}^T Z \mathbf{1} + 2 \operatorname{Tr}(nZ)^{1/2} - n \\ &= -\max_l n \|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2 - \|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} \mathbf{1}\|^2 + 2n \operatorname{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - n \\ &= -\max_l n \|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2 + 2n \operatorname{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - 2n \\ &= \underline{R}, \end{aligned}$$

where we use $(G + \mathbf{1}\mathbf{1}^T/n)^{-1} \mathbf{1} = \mathbf{1}$ in the second line.

Let η denote the difference between this lower bound \underline{R} and the value of R_{tot} achieved by the conductance vector g . This is a *duality gap* associated with g , i.e., an upper bound on the suboptimality of g . Using $R_{\text{tot}} = n \operatorname{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - n$, we can express this duality gap as

$$\begin{aligned} \eta &= R_{\text{tot}} - \underline{R} \\ &= -n \operatorname{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} + n + \max_l n \|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2 \\ &= -R_{\text{tot}} + \max_l \left(-\frac{\partial R_{\text{tot}}}{\partial g_l} \right) \\ &= -\min_l \left(\frac{\partial R_{\text{tot}}}{\partial g_l} + R_{\text{tot}} \right). \end{aligned}$$

In summary, we have the following inequality: given any feasible g , its associated total effective resistance R_{tot} satisfies

$$(33) \quad R_{\text{tot}} - R_{\text{tot}}^* \leq -\min_l \left(\frac{\partial R_{\text{tot}}}{\partial g_l} + R_{\text{tot}} \right).$$

When $g = g^*$, the right-hand side is zero by our optimality condition (28). This shows that the duality gap converges to zero as g converges to g^* .

We now derive another formulation of the dual ERMP (32). Recall that the optimal point for this dual problem is

$$Z^* = n(X^*)^{-2} = n(G^* + \mathbf{1}\mathbf{1}^T/n)^{-2},$$

where $G^* = A \operatorname{diag}(g^*) A^T$. Since $(G + \mathbf{1}\mathbf{1}^T/n)^{-2} \mathbf{1} = \mathbf{1}$, we have $Z^* \mathbf{1} = n\mathbf{1}$. Therefore, there exists an optimal dual solution satisfying $Z \mathbf{1} = n\mathbf{1}$. It follows that we can append this constraint to the dual (32) without changing its optimal value (which is R_{tot}^*). Thus we obtain the modified dual problem

$$\begin{aligned} &\text{maximize} && -\nu - (1/n)\mathbf{1}^T Z \mathbf{1} + 2 \operatorname{Tr}(nZ)^{1/2} - n \\ &\text{subject to} && a_l^T Z a_l \leq \nu, \quad l = 1, \dots, m, \\ &&& Z \succeq 0, \quad Z \mathbf{1} = n\mathbf{1}. \end{aligned}$$

We will change variables to use $Y = Z - \mathbf{1}\mathbf{1}^T$ in place of Z . The constraints on Z imply that $Y \mathbf{1} = 0$, $Y \succeq 0$, and $a_l^T Y a_l = a_l^T Z a_l \leq \nu$; conversely, if these are satisfied,

then Z is feasible in the modified problem above. We have $\mathbf{1}^T Z \mathbf{1} = \mathbf{1}^T Y \mathbf{1} + n^2 = n^2$, and, using

$$Z^{1/2} = (Y + \mathbf{1}\mathbf{1}^T)^{1/2} = Y^{1/2} + n^{-1/2}\mathbf{1}\mathbf{1}^T,$$

we have $\mathbf{Tr}(nZ)^{1/2} = n^{1/2}\mathbf{Tr} Y^{1/2} + n$. Thus, our modified dual problem can be expressed as

$$\begin{aligned} & \text{maximize} && -\nu + 2n^{1/2}\mathbf{Tr} Y^{1/2} \\ & \text{subject to} && a_l^T Y a_l \leq \nu, \quad l = 1, \dots, m, \\ & && Y \succeq 0, \quad Y\mathbf{1} = 0. \end{aligned}$$

Now let $W = Y/\nu$, so our problem becomes

$$\begin{aligned} & \text{maximize} && -\nu + 2(n\nu)^{1/2}\mathbf{Tr} W^{1/2} \\ & \text{subject to} && a_l^T W a_l \leq 1, \quad l = 1, \dots, m, \\ & && W \succeq 0, \quad W\mathbf{1} = 0. \end{aligned}$$

We can analytically maximize over ν to get $\nu = n(\mathbf{Tr} W^{1/2})^2$. With this value of ν , the objective function becomes $n(\mathbf{Tr} W^{1/2})^2$, as we get the problem

$$\begin{aligned} & \text{maximize} && n(\mathbf{Tr} W^{1/2})^2 \\ & \text{subject to} && a_l^T W a_l \leq 1, \quad l = 1, \dots, m, \\ & && W \succeq 0, \quad W\mathbf{1} = 0. \end{aligned}$$

Finally, changing variables using $V = W^{1/2}$, we can write our dual problem as

$$(34) \quad \begin{aligned} & \text{maximize} && n(\mathbf{Tr} V)^2 \\ & \text{subject to} && \|V a_l\| \leq 1, \quad l = 1, \dots, m, \\ & && V \succeq 0, \quad V\mathbf{1} = 0. \end{aligned}$$

In summary, the optimal value of the ERMP (31) is equal to the optimal value of the alternative dual problem (34). Indeed, the optimal point for (34) is given by

$$V^* = \frac{1}{\max_l \|G^{\star\dagger} a_l\|} G^{\star\dagger},$$

where $G^* = A \text{diag}(g^*) A^T$.

We can also obtain a duality gap from (34) given any feasible g by setting $V = (1/\max_l \|G^\dagger a_l\|)G^\dagger$ in (34) (clearly this V is dual feasible). It is interesting to note that the duality gap obtained with this pair of primal and dual variables, $\hat{\eta}$, is always better than that obtained in (33), except at g^* , when they are both zero:

$$(35) \quad \begin{aligned} \hat{\eta} &= n \mathbf{Tr} G^\dagger - \frac{n \mathbf{Tr} G^{\dagger 2}}{\max_l \|G^\dagger a_l\|^2} \\ &= \frac{R_{\text{tot}}}{-\min_l \partial R_{\text{tot}}/\partial g_l} \left(-\min_l \frac{\partial R_{\text{tot}}}{\partial g_l} - R_{\text{tot}} \right) \\ &= \frac{R_{\text{tot}}}{-\min_l \partial R_{\text{tot}}/\partial g_l} \eta. \end{aligned}$$

If $R_{\text{tot}} \geq -\min_l \partial R_{\text{tot}}/\partial g_l$ for a feasible g , then g is the optimal allocation; therefore the factor multiplying η is always less than or equal to 1, and is equal to 1 for $g = g^*$. So $\hat{\eta} \leq \eta$.

The dual problem (34) has the following geometric interpretation. Consider the cylindrical ellipsoid defined by

$$\mathcal{E} = \{z \mid \|Vz\| \leq 1\}.$$

This cylindrical ellipsoid has infinite extent in the direction $\mathbf{1}$; its projection on $\mathbf{1}^\perp$ is an ellipsoid, centered at 0, with semiaxis lengths $\sigma_i = 1/\lambda_i(V)$, $i = 1, \dots, n-1$, where $\lambda_i(V)$ are the nonzero eigenvalues of V . Thus, the dual problem (34) is to find the cylindrical ellipsoid \mathcal{E} that contains the points a_1, \dots, a_m and minimizes the harmonic mean of its (noninfinite) semiaxis lengths.

4.4. An Interior-Point Algorithm. The ERMP can be solved numerically using several methods, for example, via the SDP formulation (27), using a standard solver such as SeDuMi [27] or DSDP [3], or by implementing a standard barrier method [7, section 11.3], using the gradient and Hessian formulas given in section 2.9. In this section we describe a simple custom interior-point algorithm for the ERMP that uses the duality gap $\hat{\eta}$ derived in section 4.3. This interior-point method is substantially faster than an SDP formulation or a more generic method.

The logarithmic barrier for the nonnegativity constraint $g \geq 0$ is

$$\Phi(g) = -\sum_{l=1}^m \log g_l,$$

defined for $g > 0$. In a primal interior-point method, we minimize $tR_{\text{tot}} + \Phi$, subject to $\mathbf{1}^T g = 1$, using Newton's method, where $t > 0$ is a parameter; the solution of this subproblem is guaranteed to be at most m/t suboptimal. Our formula (35) gives us a nice bound on suboptimality,

$$\hat{\eta} = \frac{R_{\text{tot}}}{-\min_l \partial R_{\text{tot}} / \partial g_l} \left(-\min_l \frac{\partial R_{\text{tot}}}{\partial g_l} - R_{\text{tot}} \right) = R_{\text{tot}} \left(1 + \frac{R_{\text{tot}}}{\min_l \partial R_{\text{tot}} / \partial g_l} \right),$$

given any feasible g . We can turn this around and use this bound to update the parameter t in each step of an interior-point method by taking $t = \beta m / \hat{\eta}$, where β is some constant. (If $\hat{\eta} = 0$, we can stop because g is optimal.) This yields the following algorithm.

Given relative tolerance $\epsilon \in (0, 1)$, $\beta \geq 1$.

Set $g := (1/m)\mathbf{1}$.

while $\hat{\eta} > \epsilon R_{\text{tot}}$ **repeat**

1. Set $t = \beta m / \hat{\eta}$.
2. Compute Newton step δg for $tR_{\text{tot}} + \Phi$ by solving

$$\begin{bmatrix} t\nabla^2 R_{\text{tot}} + \nabla^2 \Phi & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} \delta g \\ \nu \end{bmatrix} = - \begin{bmatrix} t\nabla R_{\text{tot}} + \nabla \Phi \\ 0 \end{bmatrix}.$$
3. Find step length s by backtracking line search [7, section 9.2].
4. Set $g := g + s\delta g$.

When the algorithm exits, we have $R_{\text{tot}} - R_{\text{tot}}^* \leq \hat{\eta} \leq \epsilon R_{\text{tot}}$, which implies that

$$\frac{R_{\text{tot}} - R_{\text{tot}}^*}{R_{\text{tot}}^*} \leq \frac{\epsilon}{1 - \epsilon}.$$

Thus, the algorithm computes a conductance vector guaranteed to be no more than $\epsilon/(1 - \epsilon)$ suboptimal.

This algorithm differs from the standard barrier method in [7, section 11.3] in two ways: the exit condition uses the duality gap $\hat{\eta}$ from (35), and the parameter t is updated in every step of the interior-point method using $\hat{\eta}$. Using $\beta = 1$ and relative tolerance $\epsilon = 0.001$, we have found the algorithm to be very effective, never requiring more than 20 or so steps to converge for the many graphs we tried. The main computational effort is in computing the Newton step (i.e., step 2), which requires $O(m^3)$ arithmetic operations if no structure in the equations is exploited. For graphs with no more than around $m = 2000$ edges, the algorithm is quite fast. While we do not prove convergence of this algorithm, we note that the algorithm obtained by modifying the barrier method in [7, section 11.3] to use the duality gap $\hat{\eta}$ in the exit criterion is also quite effective (typically needing 30 or so Newton steps), and is provably convergent (using the same method as in [7, section 11.3.3]).

5. Analytical Solutions. In this section we describe several families of graphs for which we can solve the ERMP analytically.

5.1. Trees. When the graph is a tree, the ERMP can be expressed as

$$\begin{aligned} & \text{minimize} && \sum_{l=1}^m b_l/g_l \\ & \text{subject to} && \mathbf{1}^T g = 1, \quad g \geq 0, \end{aligned}$$

using formula (5) given in section 2.3 for R_{tot} in a tree. Here b_l is the total number of paths that pass over the edge l , which is the same as $n_l(n - n_l)$, where n_l is the number of nodes on one side of edge l . Evidently we will have $g_l^* > 0$, so, from the optimality conditions, we have $\partial R_{\text{tot}}/\partial g_l = -R_{\text{tot}}$ for all edges. This gives $b_l/(g_l^*)^2 = R_{\text{tot}}^*$, so $g_l^* = (b_l/R_{\text{tot}}^*)^{1/2}$, i.e., the optimal conductance across edge l is proportional to $b_l^{1/2}$. From $\mathbf{1}^T g^* = 1$, we find that the optimal conductance allocation is

$$(36) \quad g_l^* = \frac{b_l^{1/2}}{\sum_{k=1}^m b_k^{1/2}}.$$

The associated optimal value of the total effective resistance is

$$(37) \quad R_{\text{tot}}^* = \left(\sum_{l=1}^m b_l^{1/2} \right)^2.$$

(Note that $b_l^{1/2}$ is the geometric mean of the number of nodes on the two sides of edge l .)

As an example, consider a path on n nodes with edge k between nodes k and $k + 1$ for $k = 1, \dots, n - 1$. Then $b_k = k(n - k)$, so the optimal conductance is

$$g_k^* = \frac{(k(n - k))^{1/2}}{\sum_{j=1}^{n-1} (j(n - j))^{1/2}},$$

and the optimal total effective resistance is

$$R_{\text{tot}}^* = \left(\sum_{k=1}^{n-1} (k(n - k))^{1/2} \right)^2.$$

Using the approximation

$$\sum_{k=1}^{n-1} (k(n - k))^{1/2} \approx cn^2, \quad c = \int_0^1 (t(1 - t))^{1/2} dt \approx 0.392,$$

we have

$$g_k^* \approx \frac{(k(n-k))^{1/2}}{cn^2}, \quad R_{\text{tot}}^* \approx c^2 n^4.$$

We will see in section 6 that a path has the largest value of R_{tot}^* among all graphs with n nodes.

Our next example is a star graph with $n+1$ nodes and n edges. Each edge has one node on one side and n nodes on the other side, so $b_l = n$ for $l = 1, \dots, n$. Therefore we have $g_l^* = 1/m$, i.e., uniform allocation of conductance is optimal. (This also follows from edge symmetry, discussed below.) The optimal total effective resistance for a star graph is $R_{\text{tot}}^* = n^3$.

Our last example is a full binary tree with L levels, with a single (root) node at level 1, two nodes at level 2, and so on. Altogether there are $n = 2^L - 1$ nodes and $m = n - 1 = 2^L - 2$ edges. The two edges that connect the level 1 (root) node and the two level 2 nodes each have 2^{L-1} nodes below it, and therefore $2^{L-1} - 1$ nodes on the other side (towards the root node). In general, 2^{i-1} edges connect level $i-1$ nodes to level i nodes; each of these has $2^{L-i+1} - 1$ nodes below it and therefore $2^L - 2^{L-i+1}$ nodes on the other side. Thus, for edges that connect level $i-1$ nodes and level i nodes, we have

$$b_l = (2^{L-i+1} - 1)(2^L - 2^{L-i+1}).$$

There are 2^{i-1} such edges, for each i between 2 and L , so

$$R_{\text{tot}}^* = \sum_{i=2}^L 2^{i-1} (2^{L-i+1} - 1)^{1/2} (2^L - 2^{L-i+1})^{1/2}.$$

Using $2^{j-1} \leq 2^j - 1 < 2^j$ and $n = 2^L - 1$, we can show that

$$\frac{1}{2} \frac{(n+1)^2}{(\sqrt{2}-1)^2} \left(\sqrt{\frac{n+1}{2}} - 1 \right)^2 \leq R_{\text{tot}}^* < 2 \frac{(n+1)^2}{(\sqrt{2}-1)^2} \left(\sqrt{\frac{n+1}{2}} - 1 \right)^2.$$

So R_{tot}^* scales as n^3 for a full binary tree with n nodes.

5.2. Symmetry. We can exploit symmetry to reduce the size of the optimization problem that needs to be solved to find the optimal g . An automorphism of a graph is a permutation π of the nodes such that $(i, j) \in E$ if and only if $(\pi(i), \pi(j)) \in E$. An automorphism induces a permutation of the edges as well: it maps edge l to edge k if the nodes adjacent to edge l are mapped under π to the nodes adjacent to edge k . We say that two edges are *symmetric* if there is an automorphism of the graph that maps one edge to the other. The basic result we will use is this: If two edges are symmetric, they must have the same optimal conductance value.

To show this, consider the graph with a set of edge conductances g , and let π be an automorphism of the graph. We form a new conductance vector \hat{g} as follows. If edge l is mapped to edge k by π , then we take $\hat{g}_k = g_l$. In other words, we permute the edge conductance values according to the edge permutation induced by π . Let $G = A \text{diag}(g) A^T$ and $\hat{G} = A \text{diag}(\hat{g}) A^T$. Then we have

$$\hat{G} = \Pi G \Pi^T,$$

where Π is the permutation matrix representation of π . It follows that $R_{\text{tot}}(g)$ and $R_{\text{tot}}(\hat{g})$ are equal, since \hat{G} and $\Pi G \Pi^T$ have the same eigenvalues. Finally, suppose

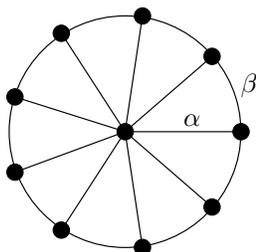


Fig. 1 A wheel graph on 10 nodes. The conductance of each spoke is α , and the conductance of each rim edge is β .

that g is optimal. Then since \hat{g} has the same total effective resistance, it is also optimal. But the optimal conductance is unique, so we must have $\hat{g} = g$. This means that if an edge is mapped to another edge, i.e., the pair of edges is symmetric, then they must have the same optimal conductance value. (For more on symmetries in convex optimization in general, see, e.g., [7, Ex. 4.4]; for more in the context of graph optimization problems, see [5, 18].)

A graph is *edge-transitive* if all pairs of edges are symmetric, i.e., for any two edges, there is an automorphism of the graph that maps one edge to the other. For an edge-transitive graph, the optimal conductances are all equal, i.e., we have $g_l^* = 1/m$, $l = 1, \dots, m$. Examples of such graphs include ring graphs, star graphs, and hypercube graphs.

As a less trivial example of exploiting symmetry, we consider the wagon wheel graph on $n + 1$ nodes, shown in Figure 1. Any pair of spoke edges is symmetric, and any pair of rim edges is symmetric (for example, by the automorphism that rotates one spoke to the other). By symmetry, the optimal conductances on the spoke edges are all the same, say, α , and the optimal conductances on all the rim edges are also all the same, say, β .

Thus the optimal conductance matrix $G = A \text{diag}(g^*) A^T$ has the form

$$G = \begin{bmatrix} n\alpha & -\alpha \mathbf{1}^T \\ -\alpha \mathbf{1} & \alpha I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \beta L_{C_n} \end{bmatrix},$$

where L_{C_n} is the conductance matrix with unit weights for the ring graph C_n on n nodes. The eigenvalues of G can be shown to be

$$0, \quad (n+1)\alpha, \quad (\alpha + 2\beta(1 - \cos(2\pi k/n))), \quad k = 1, \dots, n-1.$$

Thus, the ERMP becomes the problem

$$(38) \quad \begin{aligned} & \text{minimize} && 1/(n+1)\alpha + \sum_{k=1}^{n-1} 1/(\alpha + 2\beta(1 - \cos(2\pi k/n))) \\ & \text{subject to} && n(\alpha + \beta) = 1, \quad \alpha \geq 0, \quad \beta \geq 0, \end{aligned}$$

with variables α and β . We can eliminate one of the variables, say, α , and set the derivative to zero to see that the optimal β must satisfy

$$\sum_{k=1}^{n-1} \frac{1 - 2\cos(2\pi k/n)}{(1 + n\beta^*(1 - 2\cos(2\pi k/n)))^2} = \frac{1}{n+1} \frac{1}{(1 - n\beta^*)^2}.$$

There is exactly one solution of this equation, since problem (38) is strictly convex. We can find β^* numerically using a simple bisection method or Newton's method.

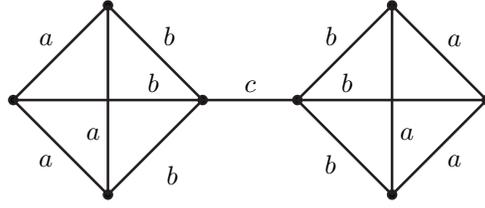


Fig. 2 A barbell graph on 8 nodes.

The solution is quite interesting: for $5 \leq n \leq 10$, more conductance is assigned to the edges than the rim, but for $n \geq 11$, more conductance is assigned to the spokes than to the rim. (When $n = 4$, the wheel graph is the same as the complete graph, so $\alpha^* = \beta^*$.)

As another example, we consider the barbell graph $K_n - K_n$ on $2n$ nodes, which consists of two fully connected components of size n joined by a single edge between nodes n and $n+1$, as shown in Figure 2. By symmetry, there are exactly three distinct weights: the weights on edges neither of whose endpoints is n or $n+1$, a ; the weights on edges with exactly one endpoint n or $n+1$, b ; and the weight on the edge between n and $n+1$, c . The conductance matrix G for these weights is

$$G = \begin{bmatrix} \alpha I - a\mathbf{1}\mathbf{1}^T & -b\mathbf{1} & & & \\ -b\mathbf{1} & \gamma & -c & & \\ & -c & \gamma & -b\mathbf{1} & \\ & & -b\mathbf{1} & \alpha I - a\mathbf{1}\mathbf{1}^T & \\ & & & & \end{bmatrix},$$

where $\mathbf{1} \in \mathbf{R}^{n-1}$, $\alpha = a(n-1) + b$, and $\gamma = (n-1)b + 2c$.

The $2n$ eigenvalues of the conductance matrix with these weights can be shown to be

$$\begin{aligned} &0, & a(n-1) + b & \text{with multiplicity } 2n - 4, \\ &nb, & (1/2)(nb + 2c \pm ((2c + nb)^2 - 8bc))^{1/2}. \end{aligned}$$

Therefore, we have reduced the ERMP problem (which has $m = n(n-1)+1$ variables) to the problem

$$(39) \quad \begin{aligned} &\text{minimize} && (2n-4)/(a(n-1)+b) + 1/nb + (nb+2c)/2bc \\ &\text{subject to} && (n-1)(n-2)a + 2(n-1)b + c \leq 1, \\ &&& a, b, c \geq 0, \end{aligned}$$

which has three variables: a , b , and c .

This problem has an analytical solution: the optimal weights are

$$a^* = \frac{\mu}{n-1} \left(\sqrt{2} - \frac{\sqrt{n+1}}{n} \right), \quad b^* = \mu \frac{\sqrt{n+1}}{n}, \quad c^* = \mu \sqrt{\frac{n}{2}},$$

where μ is a normalizing constant,

$$\mu = \left((n-2) \left(\sqrt{2} - \frac{\sqrt{n+1}}{n} \right) + 2(n-1) \frac{\sqrt{n+1}}{n} + \sqrt{\frac{n}{2}} \right)^{-1}.$$

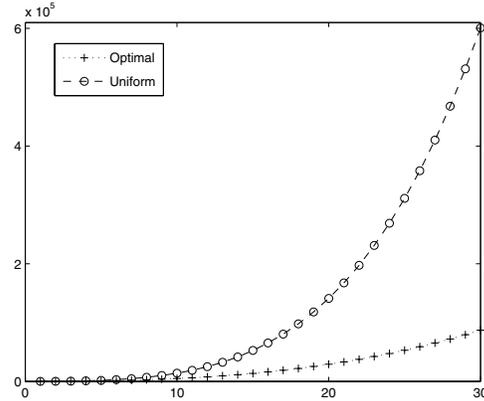


Fig. 3 The optimal value R_{tot}^* , and R_{tot} with uniform weights $1/m$, as a function of n for the barbell graph.

Evidently R_{tot}^* scales as n^3 for the barbell graph $K_n - K_n$. For the same graph, the total effective resistance obtained with uniform weights $g_l = 1/m$ scales as n^4 . We conclude that the suboptimality of the uniform weights grows unboundedly with n , i.e., the optimal conductances are unboundedly better than uniform conductances. In Figure 3, the optimal R_{tot}^* is plotted as a function of n for the barbell graph along with the total effective resistance with uniform weights.

6. Bounds on Optimal Total Effective Resistance. In this section we show that among all graphs on n nodes, the path (with $m = n - 1$ edges) has the largest value of R_{tot}^* and the complete graph (with $m = n(n - 1)/2$ edges) has the smallest value of R_{tot}^* . That is, for any graph on n nodes,

$$(40) \quad n(n - 1)^2/2 \leq R_{\text{tot}}^* \leq \left(\sum_{k=1}^n (k(n - k))^{1/2} \right)^2.$$

The left-hand term is R_{tot}^* for the complete graph K_n , and the right-hand term is R_{tot}^* for the path on n nodes. The right-hand term grows as n^4 , so the ratio grows as n . The result (40) makes sense: it states that the path is the “least connected” graph and the complete graph is the “most connected” graph, when measured by optimal total effective resistance. (We note that [24] shows that the total effective resistance with *uniform* rather than *optimal* weights is largest for the path and smallest for the complete graph.)

We first observe that the optimal value of the ERMP, R_{tot}^* , cannot increase when edges are added to the underlying graph \mathcal{N} , since any allocation of conductance on the original graph is also feasible for the graph when edges are added. In other words, the optimal total effective resistance is monotone nonincreasing in E , the set of edges of the graph. (It is *not* monotone nonincreasing in the number of edges, $|E|$.)

It follows that the smallest value of R_{tot}^* , among graphs on n nodes, is obtained for the complete graph K_n . By symmetry, the optimal allocation of conductances for K_n is uniform. Thus, the optimal conductance matrix is

$$A \text{diag}(g^*)A^T = G^* = (n/m)(I - \mathbf{1}\mathbf{1}^T/n),$$

where $m = n(n - 1)/2$. The eigenvalues of this matrix are 0 and $n/m = 2/(n - 1)$

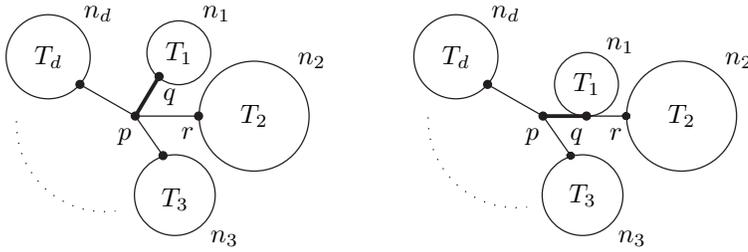


Fig. 4 The tree on the right, \hat{T} , is formed from T , the tree on the left by splicing node q between p and r .

repeated $n - 1$ times. So $R_{\text{tot}}^* = n(n - 1)^2/2$ for K_n . This establishes the lower bound in (40).

We now turn to the upper bound in (40). Since R_{tot}^* cannot decrease as edges are removed from the graph, we conclude that the largest value of R_{tot}^* among graphs with n nodes is obtained for a tree. It remains to show that among all trees on n nodes, the path has the largest value of R_{tot}^* . This implies that R_{tot}^* for the path is an upper bound on R_{tot}^* for any graph on n nodes.

Let T be a tree that is not a path, i.e., a tree with at least one node with degree greater than 2. We will show that we can construct a tree \hat{T} from T such that $R_{\text{tot}}^*(\hat{T}) > R_{\text{tot}}^*(T)$. This will establish our claim that the path has the highest value of R_{tot}^* among all trees (and, therefore, among all graphs) on n nodes.

Let p be a node in T with degree $d \geq 3$. The removal of p disconnects T into d subtrees, T_1, \dots, T_d , with sizes n_1, \dots, n_d , ordered so that $n_1 \leq n_i$, $2 \leq i \leq d$. Clearly, $n_1 \leq (n - 1)/d < n/2$, since $d > 2$. Let q be the vertex that p is connected to in T_1 . Form \hat{T} from T by inserting q between p and r , where r is the vertex that p is connected to in T_2 . (See Figure 4.) It is easily verified that $b_l = n_l(n - n_l)$ does not change for any edge except the edge between p and q , since the number of nodes to one side of the edge does not change for any of these edges. The value of b_l for (p, q) is $n_1(n - n_1)$ in T and is $(n_1 + n_2)(1 + \sum_{j=3}^d n_j)$ in \hat{T} . Since $n_1 < n/2$ and $n_1 \leq n_i$, $i = 2, \dots, d$,

$$\min\{n_1, n - n_1\} = n_1 < \min\{n_1 + n_2, 1 + n_3 + \dots + n_d\}.$$

So the value of b_l for $l \sim (p, q)$ increases from T to \hat{T} :

$$b_l(T) = n_1(n - n_1) < (n_1 + n_2)(1 + n_3 + \dots + n_d) = b_l(\hat{T}).$$

Since b_l for all other edges does not change from T to \hat{T} , we can conclude from (37) that $R_{\text{tot}}^*(T) < R_{\text{tot}}^*(\hat{T})$. (Note that the choice of T_2 is arbitrary: we could have attached q between p and any other subtree; the crucial point is to move a subtree of smallest size.) We have now established our bounds (40).

We note that a similar argument can be used to show that given any tree for which there is a node at distance 2 from the node of maximum degree, a new tree can be formed which has a *smaller* value of R_{tot}^* . This means that the tree with the smallest value of R_{tot}^* is the star.

6.1. Lower Bound on Total Effective Resistance with Uniform Weights. Here we derive a simple lower bound on $R_{\text{tot}}^{\text{unif}}$, the total effective resistance with uniform weights on the edges, i.e., for $g = (1/m)\mathbf{1}$. With uniform weights, the conductance

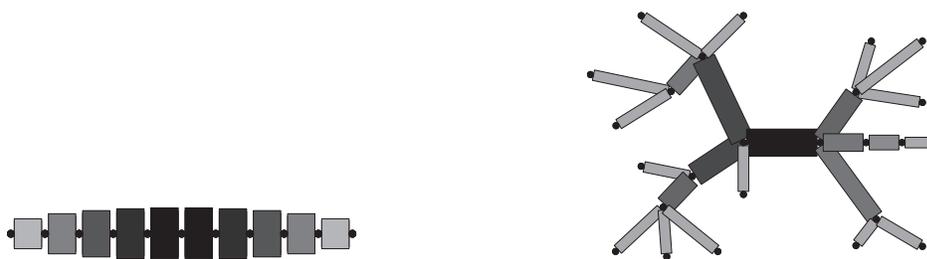


Fig. 5 *Left: Optimal conductance allocation on a path on 11 nodes. Right: Optimal conductance allocation on a tree with 25 nodes.*

matrix is

$$G^{\text{unif}} = (1/m)AA^T = (1/m)L,$$

where L is the (unweighted) Laplacian of the graph. Therefore we have

$$R_{\text{tot}}^{\text{unif}} = n \sum_{i=2}^n \frac{1}{\lambda_i(G)} \geq \frac{n}{\lambda_2(G)} = \frac{nm}{\lambda_2(L)}.$$

This shows that a graph with small algebraic connectivity must have large $R_{\text{tot}}^{\text{unif}}$. We can bound $\lambda_2(L)$, the algebraic connectivity of the graph, in terms of the size of a cut in the graph [13, 18]:

$$\lambda_2(L) \leq \min_{X \subseteq V} \frac{n|E_X|}{|X||X_c|},$$

where $|X|$ is the size of a subset of nodes X , X_c is the set of remaining nodes, and $|E_X|$ is the number of edges in the cut that partitions the graph into (X, X_c) . Using this, we get

$$(41) \quad R_{\text{tot}}^{\text{unif}} \geq \max_{X \subseteq V} \frac{m|X||X_c|}{|E_x|}.$$

This bound says that among graphs on n nodes with m edges, uniform allocation of conductance leads to a large total effective resistance for graphs which have sparse cuts. The barbell graph is a graph with one very sparse cut; it is therefore not surprising that the uniform allocation is much worse than the optimal allocation of weights for the barbell.

7. Examples. In this section we show some examples of optimal conductance allocations on graphs. In each example, we draw the edges with width and color saturation proportional to the optimal edge conductance.

Our first two examples are a path on 11 nodes and a tree on 25 nodes, shown in Figure 5. The optimal conductances for these come from (36). The optimal conductance is larger on edges with more paths passing through them than edges near the leaves, which have fewer paths passing through them.

Our next two examples are an 8×8 mesh and a modified 8×8 mesh, shown in Figure 6. The optimal conductances for these graphs were computed numerically.

In Figure 7, we plot the optimal conductances for a barbell. Finally, in Figure 8, we plot the optimal conductances for a randomly generated graph with 25 nodes and 88 edges. Here too we see large conductance allocated to edges across sparse cuts.

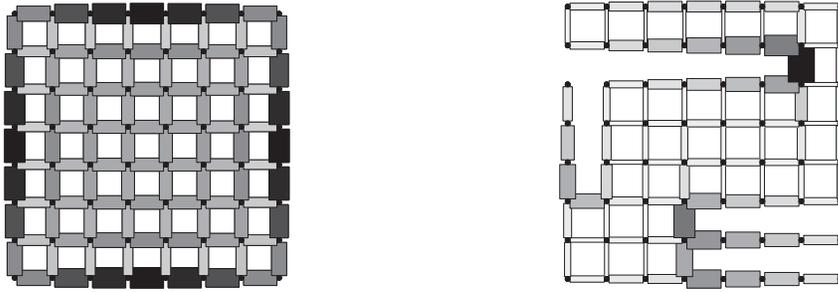


Fig. 6 Optimal conductance allocation on an 8×8 mesh (left) and on a modified 8×8 mesh (right).

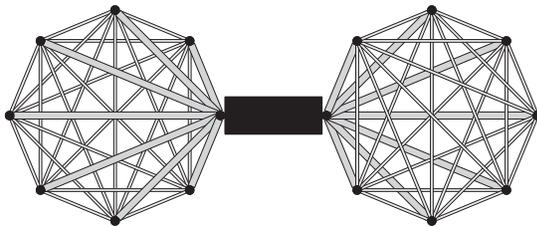


Fig. 7 Optimal conductance allocation on barbell graph with 16 nodes.

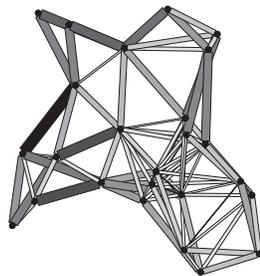


Fig. 8 Optimal conductance allocation on randomly generated graph with 25 nodes and 88 edges.

8. Extensions. We conclude by listing some variations on the ERMP that are convex optimization problems and can be handled using similar methods. Since each R_{ij} is a convex function of the conductances, we can minimize any nondecreasing convex function of the R_{ij} (which is convex). Some interesting (convex) objectives, in addition to R_{tot} , include the following.

- *Minimizing effective resistance between a specific pair of nodes.* We allocate conductance to minimize the effective resistance between a specific pair of nodes, i and j . This problem has the following simple solution: Allocate the conductance equally to the edges lying on a shortest path between i and j . This conductance allocation violates our assumption (4); the effective resistance between any pair of nodes not on the path is undefined.
- *Minimizing the sum of effective resistances to a specific node.* This problem can be formulated as an SDP and solved by modifications to the methods

discussed in this paper. Examples show that the optimal allocation of weights need not be a tree.

- *Minimizing maximum effective resistance.* The maximum effective resistance over all pairs of nodes, $\max_{i,j} R_{ij}$, is the pointwise maximum of the convex functions R_{ij} , and is therefore also a convex function of the conductances g [7, section 3.2.3]. The problem of minimizing the maximum effective resistance is a convex optimization problem, can be formulated as an SDP, and solved using standard interior-point methods.

We mention one more extension, minimizing R_{tot} without the nonnegativity constraints on the edge conductances:

$$(42) \quad \begin{array}{ll} \text{minimize} & R_{\text{tot}} \\ \text{subject to} & \mathbf{1}^T g = 1. \end{array}$$

In this problem the conductances can be negative, but we restrict the domain of the objective R_{tot} to $\{g \mid G + \mathbf{1}\mathbf{1}^T/n \succ 0\}$. This problem can be solved using Newton's method, using the derivatives found in section 2.9. The optimality conditions for this problem are simply that $\mathbf{1}^T g = 1$ (feasibility) and that all components of the gradient, ∇R_{tot} , are equal; specifically, $\nabla R_{\text{tot}}(g) = -R_{\text{tot}}\mathbf{1}$.

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