Christmas Gift Exchange Games

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Abstract. The Christmas gift exchange is a popular party game played around Christmas. Each participant brings a Christmas present to the party, and a random ordering of the participants, according to which they will choose gifts, is announced. When a participant's turn comes, she can either open a new gift with unknown value, or steal an already opened gift with known value from someone before her in the ordering; in the second case, the person whose gift was stolen gets to make the same choice.

We model the gift exchange as a sequential game of perfect information and characterize its equilibria, showing that each player plays a *threshold* strategy in the subgame perfect equilibrium of the game. We compute the expected utility of players as a function of the position in the random ordering; the first player's utility is vanishingly small relative to every other player. We then analyze a different version of the game, also played in practice, where the first player is allowed an extra turn after all presents have been opened—we show that all players have equal utility in the equilibrium of this game.

1 Introduction

The practice of giving gifts to friends and relatives at Christmas is a centuries-old tradition. Contrary to popular belief, gifts are not conjured up by Santa at the North Pole, and the actual buying of gifts unfortunately must be done by real people in real local marketplaces with no aid from Santa whatsoever. As a result, the Christmas gift industry is now a huge multibillion dollar industry much larger than online advertising, and is therefore clearly a subject deserving of serious study.

The problem of Christmas gifting admits many interesting directions of research drawing on various disciplines. For instance, the phenomenon of stores beginning to play Christmas jingles annoyingly earlier and earlier in the year (to subtly indicate gift-shopping time) is related to the phenomena of unraveling markets studied in the economics literature [5]. Another obvious area is psychology and sociology, analyzing the very noticeable impact of gifting and associated customs on the behavior of otherwise normal individuals. We will, however, focus on a particular Christmas gifting tradition, the *gift exchange*, that leads to increased welfare and significant computational and storage savings for each gifter: instead of buying one gift individually for each giftee, every gifter brings a single gift, which are then collectively exchanged amongst the group.

The gift exchange mechanism represents a significant improvement for gifters over the tradition of buying individual gifts, since only 1 present need be bought rather than n — that is, the number of gift-choosing problems that need to be solved, and the fear struck by the prospect of Christmas gift shopping, is now reduced to a constant rather than growing linearly with the size of an player's neighborhood in the Christmas-gifting-graph[1]. This also has the second-order benefit of removing the incentive to limit the size of one's social network due to the large psychological and economic gifting costs, which in the limit could lead to smaller and smaller tight-knit groups, eventually resulting in a catastrophic unraveling of society. The gift exchange scheme also improves *efficiency* relative to the scheme of one gift per neighboring node used widely in practice. Empirical and anecdotal evidence suggests that most people, including the President [1], believe they give better gifts than they get; the value obtained from the average received gift is much lower than the sum of the costs of planning to buy, buying, and giving the gift (and recovering from the experience). Since each gifting transaction leads to negative utility on average, welfare is maximized by minimizing the number of transactions— this means that, subject to the constraint that each person must receive a gift, the gift exchange mechanism actually has optimal welfare.[2]

The protocol used to assign the n gifts amongst the n participants is known by several different names including Chinese Christmas and White Elephant gift exchange, and is a common party game around Christmas time[2–4]. Each participant brings one gift (whose value is in some range prespecified by the host) to the party, and the gifts are all placed in a pile (presumably under a suitably large and well-decorated Christmas tree). We'll assume that everyone uses identical wrapping, so that they cannot identify the gift they brought once it's been put in the pile; the reason for this assumption will become clear once we present our model. A random ordering of the participants is chosen, and when a participant's turn comes, she either opens a new gift, or steals an already opened gift from someone before her in the ordering, in which case the person whose gift was stolen gets to make the same choice. A person cannot steal back the gift that was stolen from her immediately; also, to ensure that the last person is not guaranteed to walk away with the best present, a rule that no present can be stolen more than a certain number of times, say three or four times, is enforced.

1 This is based on the fairly reasonable assumption that the number of distinct Christmas parties an player must attend and buy gifts for is 1 (or very small), that is, it does not scale with the size of his Christmas-gifting neighborhood.

2 There is an interesting parallel between the VCG mechanism and the gift exchange scheme — both mechanisms have excellent efficiency properties, but are nonetheless not popular in the industry. While the reason for this is very obvious for the gift exchange scheme (the gift producing industry has no reason to like this scheme which leads to fewer gifts being bought), the case for the VCG auction is far more interesting and subtle, see The Lovely but Lonely Vickrey Auction [6].

In this paper, we study this gift exchange mechanism from a game theoretic point of view. Since the mechanism itself is rooted in tradition, we do not address the question of whether or not this is a good mechanism to fairly distribute the gifts. Rather, we analyze the game from the participants perspective, and investigate best response— given the gift exchange mechanism, how best should a utility maximizing partygoer behave? (We mean best behave not in the sense of being on one's best behavior at the family party, but rather from the point of view of maximizing the expected value of the final present she goes home with). To do this, we model the Christmas party gift exchange as a sequential game where each gift's value is drawn uniformly at random and unknown until unwrapped, and use backward induction to reason about the behavior of each player in this game. The problem becomes technically interesting because in addition to not knowing the value of an unopened present, an player also has to contend with the fact that his present may be stolen away from him in the future, depending on its desirability.

The remainder of the paper is organized as follows. We first present a formal model for the gift exchange game in §2, and then derive the best response and equilibrium in §3. Our results allow us to quantify the benefit of drawing a position towards the end of the list of participants, answering the question of how loudly to sigh or squeal when learning of one's position in the random order. Finally, we analyze a version of the game with slightly modified rules where the first player gets a chance to steal a present at the end if she has not done so already; this version is also sometimes played in practice. In contrast with the original version, all players have equal expected utility in the equilibrium of this game.

2 Model

There are n players, and n unopened gifts. The gifts have a common value to each player, that is, different players do not value the same gift differently (for instance, this common value could be exactly the dollar value of the gift). Each gift has a value v_i drawn independently and uniformly at random from $[0, 1]$. The value of a gift remains unknown until the gift is opened, at which point its value v_i becomes known to all n players.

A random ordering is chosen amongst the n players and announced publicly— this is the order in which players get to choose presents. We number the players according to this order. We also adopt the rule from the actual party game that each gift can be stolen only a limited number of times— we restrict the number of times a gift can be stolen to 1, i.e., if a present has been stolen once, it cannot be stolen again, and stays with its current owner. We will call a present that has been opened but never stolen *available*, and an open present that has been stolen once already *unavailable*. Limiting the number of times a present can be stolen to 1 keeps the analysis tractable while still preserving the feature that an player's current choice affects the future decision of whether other players will want to steal her gift in the future, affecting her final value.

When player i's turn arrives, she has a choice between picking an unopened present (with unknown value drawn UAR $[0, 1]$) from the pile, or stealing an available present from the players $1, \ldots, i-1$. If she opens a new gift, the game proceeds to player $i+1$. If she steals an open available gift from some player $j < i$, j again gets a choice between stealing an available present and opening a new gift from the pile (note that j cannot steal back her own gift since it has been stolen once and is now unavailable). Note that when player i's turn arrives, there are exactly $n - (i - 1)$ unopened gifts in the pile, and each player $1, ..., i - 1$ has an opened gift. The game continues this way until there is only one unopened gift, at which point player n takes her turn and follows the same sequence. Define a step to mean each time a gift has a new owner: the game is guaranteed to terminate in at most $2n$ steps, since there are n gifts, and each gift can be stolen only once, corresponding to at most 2 steps per gift in the game.

We analyze the gift exchange game G as a sequential game with perfect information, where the players are rational utility maximizers— each player tries to maximize the expected value of the final gift she is left with, given that each player after her is a rational utility maximizer as well (the expectation is taken over the random draws of unopened gifts). We point out that our model assumes that players only value gifts and not time, and does not address players who are running out of patience (or lacked it to start with), or want to get the gift exchange over with quickly. These can be modeled with a discount factor; we leave this as an open direction for future work. Our results also only apply when there are no externalities— they do not, for instance, predict the outcome of a game where your coworker five places down the line might steal your present either out of love for the present, or hate for you. While these assumptions are common in the research literature, they (especially rationality) may not hold in practice — to ensure applicability in practice, it is adequate to have your fellow partygoers read and understand the best response derived in §3, and instruct them to act according to it, before starting the gift exchange.

3 Analysis of the Game

In this section, we analyze the equilibrium of the gift exchange game. Before beginning with the analysis, we first make some simple observations about the game. We define round i as the sequence of steps starting from when player i first gets a turn to the step immediately before player $i+1$ first gets her turn. Note that a new gift is opened in the last step of a round (a round can have only one step), and exactly one gift is opened in each round. Round i has no more than i steps, and the entire game terminates in no more than $2n$ steps. The last player plays exactly once; the player in the ith position in the ordering plays at most $n - i + 1$ times. Once a player steals a gift, she never plays another turn, and the value of the gift she steals is her final value from the game.

We now give a complete analysis of the game G. We prove that in the solution of the game, each player plays according to a *threshold strategy* of the following form: if the value of the most valuable available gift is at least θ (where θ is the threshold), then steal that gift; otherwise, open an unopened gift. The value of the threshold θ depends on which round in the game is in progress—specifically, θ is a function of the number i of *unopened gifts* at the time. We define a sequence $\theta_1, \ldots, \theta_n$ recursively as follows:

$$\theta_1 = 1/2, \qquad \theta_i = \theta_{i-1} - \frac{\theta_{i-1}^2}{2} \text{ for } i > 1. \tag{1}$$

Note that this recurrence defines a decreasing sequence. We prove the following result.

Theorem 1. *The following is a subgame perfect equilibrium of the gift exchange game: for any player p and any time p gets to play, play the threshold strategy with threshold θ_i, where i is the number of unopened gifts at the time. Furthermore, the expected value that p receives by playing this strategy is equal to* $\max(\theta_i, v)$*, where v is the value of the most valuable available gift at the time.*

Proof. We prove this by induction on i. We start with $i = 1$. This means that at the time player p gets to play, only one unopened present is left. This player has two choices: either to steal the most valuable available gift (of value v), or to open the only remaining unopened gift, after which the game will end. Since the value of a gift is drawn uniformly from $[0, 1]$, the expected value of opening the unopened gift is $1/2$. Thus, the player must steal if $v \geq 1/2$ and open the unopened gift if $v < 1/2$. The value that this strategy gets is precisely $\max(1/2, v)$.

Now, we assume the statement is proved for $i-1$, and prove it for i. Consider the player p that is playing at a time that there are exactly i unopened gifts, and the value of the most valuable available gift is v. At this point, p has two options: either to steal the gift of value v, or to open an unopened gift. If p opens an unopened gift, we denote the value of this gift by x, drawn uniformly from $[0, 1]$. In the next step, by the induction hypothesis, the next player will steal the highest value available gift if this gift has value at least θ_{i-1}. If she does so, the player whose gift is just stolen will get to play, and again, by the induction hypothesis, will steal the highest value avaliable gift if its value is at least θ_{i-1}. This ensues a sequence of stealing the highest value available gifts, until we reach a point that the highest value available gift has value less than θ_{i-1}, at which point the person whose turn it is to play will open a new gift.

We now consider two cases: either the value x of the gift p just opened is at least θ_{i-1}, or it is less than θ_{i-1}. In the former case, the sequence of stealings will at some point include p. At this point, all the available gifts of value more than x are already stolen. We denote by $v(x)$ the highest value of an available gift of value less than x. This is precisely the value of the most valuable gift that is still available at the time that the sequence of stealings reaches p. By induction hypothesis, at this point, the maximum value that p gets is equal to $\max(\theta_{i-1}, v(x))$. In expectation, the value of p in this case is $\max(\theta_{i-1}, \mathrm{Exp}[v(x)])$, where the expectation is over drawing x uniformly at random from $[\theta_{i-1}, 1]$.

The other case is when x is less than θ_{i-1}. In this case, since the sequence θ is decreasing, by induction hypothesis the gift that p just opened will never be stolen. Therefore, the expected value of the gift that p ends up with in this case is precisely $\theta_{i-1}/2$.

Putting these together, the overall value of p, if she decides to open a new gift can be written as

$$(1-\theta_{i-1})\max(\theta_{i-1}, \mathrm{Exp}_{x\leftarrow U[\theta_{i-1},1]}[v(x)]) + \theta_{i-1} \times \frac{\theta_{i-1}}{2}.$$

Now, we consider two cases: if $v > \theta_{i-1}$, then we have $v > \theta_{i-1}/2$ and $v \geq v(x)$ (the latter inequality by the definition of v and $v(x)$). Therefore, the above expression is less than $(1-\theta_{i-1})v + \theta_{i-1}v = v$, meaning that in this case, it is p's optimal strategy to steal the gift of value v. In the other case ($v \leq \theta_{i-1}$), by the definition of $v(x)$, for every $x \in [\theta_{i-1}, 1]$, $v(x) = v \leq \theta_{i-1}$. Therefore, the utility that p obtains by opening a new present can be written as

$$(1-\theta_{i-1})\theta_{i-1} + \theta_{i-1} \times \frac{\theta_{i-1}}{2} = \theta_i.$$

Putting everything together, we obtain that the maximum utility p can obtain is $\max(v, \theta_i)$, and this utility is obtained by playing the threshold strategy with threshold θ_i.

We cannot obtain an explicit formula for θ_i from the recurrence relation (1), but the following theorem gives us the asymptotics.

Theorem 2. *For every i, we have $\frac{2}{i+2+H_i} \leq \theta_i \leq \frac{2}{i+3}$, where $H_i \approx \ln(i) + \gamma$ is the i'th harmonic number.*

Proof. Let $y_i = 2/\theta_i$. The recurrence (1) gives us:

$$y_i = \frac{2}{2/y_{i-1} - (2/y_{i-1})^2/2} = \frac{y_{i-1}^2}{y_{i-1}-1} = y_{i-1} + 1 + \frac{1}{y_{i-1}-1}. \qquad (2)$$

Thus, since the term $1/(y_{i-1}-1)$ is non-negative, we have $y_i > y_{i-1}+1$, which together with $y_1 = 4$ implies that $y_i > i+3$, proving the upper bound on θ_i. To prove the lower bound, we use the inequality $y_i > i+3$ we just proved in combination with (2). This gives us $y_i < y_{i-1} + 1 + \frac{1}{i+2}$. This implies $y_i < i + 3 + \sum_{j=4}^{i+2} \frac{1}{j} < i + 2 + H_i$, proving the lower bound on θ_i.

That is, as we move along the random ordering, a player's threshold for stealing a gift keeps increasing: early in the game, players are willing to settle for gifts of lower value than later in the game (recall that if a gift is stolen, that gift's value is the final utility to the player who steals the gift).

We make the following observations about the equilibrium play of the game, which follow from the fact that the optimal strategy for each player is a threshold strategy, and these thresholds increase through the play of the game:

– If a gift is not stolen immediately after it is opened, it is never stolen, since the thresholds θ increase as the number of unopened gifts decreases.
– If a gift is stolen from a player, this player does not continue stealing, but rather opens a new gift. That is, each round is of length at most 2, *i.e.*, there are no 'chains' of gift stealing in any round. Each player i stealing a gift therefore steals either from $i-1$, if $i-1$ opened a new gift, or else from the player $j < i-1$ from whom $i-1$ stole her gift (and who consequently opened a new gift), in this case, all players $j+1, j+2, \ldots, i$ have stolen the gifts opened by j.

Therefore, when players play according to their optimal strategy (the threshold strategies prescribed by Theorem 1), the game will proceed as follows: first, player 1 opens a new present. If the value of this present is less than θ_{n-1}, this present will not be stolen by player 2 (and by no other player, since $\theta_{n-1} < \theta_i$ for $i > n-1$), and player 2 opens a new present; otherwise, this present will be stolen by player 2, and player 1 will open a new present. In either case, if the value of the newly opened gift is less than θ_{n-2}, it will not be stolen by player 3 (and therefore by no other player after that), and instead, player 3 opens a new present; but if this value is greater than θ_{n-2}, it will be stolen by player 3, and the player who used to hold that gift will open a new present, and so on. When it is turn for player i's to play for the first time, it must be that in the last step, one of the players has opened a new gift (unless $i = 1$). If the value of this gift is more than θ_{n-i+1}, player i steals it, and the player who used to hold that gift will open a new gift. Otherwise, player i opens a new gift.

Given this, we can calculate the expected utility of each player in this game: for every $i > 1$, when player i gets to play for the first time, the only way the value v of the most valuable available gift is greater than θ_{n-i+1} is if this gift is the one just opened by the last player who played before i. This happens with probability $1 - \theta_{n-i+1}$, and in this case, the value of the gift is distributed uniformly in $[\theta_{n-i+1}, 1]$. Therefore, by Theorem 1, the expected value that player i derives in this game is precisely

$$\text{Exp}[\max(\theta_{n-i+1}, v)] = (1-\theta_{n-i+1}) \times \frac{1+\theta_{n-i+1}}{2} + \theta_{n-i+1} \times \theta_{n-i+1} = \frac{1}{2} + \frac{\theta_{n-i+1}^2}{2}.$$

Therefore, all players $i > 1$ derive a utility more than $1/2$. This, however, is at the expense of the first player. When player 1 plays for the first time, there is no available gift. Therefore, by Theorem 1, the expected utility of player 1 is precisely θ_n, which by Theorem 2 is $\frac{2}{n}(1 + o(1))$. This is summarized in the following theorem.

Theorem 3. *The expected utility of player i for $i > 1$ in the gift exchange game is $\frac{1}{2} + \frac{\theta_{n-i+1}^2}{2}$. For player 1, the expected utility of playing the game is $\theta_n = \frac{2}{n}(1 + o(1))$.*

3.1 A Fairer Game

The first player in the ordering might never get to steal a gift in the game G, and as we saw above, receives very low utility relative to all other players: in this sense, the game G is not very fair. To be more fair to the first player, a version of the game is sometimes played where the first player gets a turn at the end to steal a gift. We next analyze this version of the game, and show that it is indeed more fair for the first player — every player has equal expected utility in the equilibrium of this game.

We define the game G' to be the following modification of G: if the first player never gets a chance to steal a gift through the course of the play, she gets a turn at the end after all gifts have been opened, and can steal from amongst the available gifts if she wishes. (Note that if player 1 has never stolen a gift, the gift she holds is always available; if she chooses to keep her own gift at the end of the game, we will call this equivalent to stealing her own available gift.) We show the following about G'.

Theorem 4. *In the subgame perfect equilibrium of G', every player has expected utility* $1/2$.

Proof. We claim that the following is a subgame perfect equilibrium of the modified game G'. Each player i other than player 1 uses the following strategy. If player 1 has already stolen a gift, then play according to the optimal strategy for G; if player 1 has not yet stolen a gift, steal the maximum value available gift. For player 1, if her gift is stolen when there are 2 or more unopened items, she steals the highest value available gift if its value v is greater than $1/2$, else opens a new gift. If there is only one unopened gift when her gift is stolen, she opens the new gift, and if she gets a turn when there are no unopened gifts, she steals the highest value available gift (including her own).

We prove this claim by backward induction. First, note that if player 1 does play after all gifts have been opened, she must steal the highest value gift from amongst the available gifts (including her own). If there is exactly one unopened gift when her gift is stolen, there is no player who can steal the new gift she opens from her: if the best available gift has value v_1, she can get a value of v_1 by stealing, or $\max\{v_1, x\} \geq v_1$ if she opens a new gift (since v_1 will still be available). So she must open the new gift. Also, once player 1 steals a gift, it is optimal for every player to play according to the strategy described for G in Theorem 1. Consider a player $j \neq 1$ when there is just one remaining unopened gift, when 1 has not stolen a gift yet. She can either open a new gift with value x, or steal the best available gift of value v_1. If she steals, this gift cannot be stolen from her, so her final value is v_1. If she opens a new gift with value $x > v_1$, this gift becomes the highest value available gift, and will be stolen by player 1 in the next round, leaving her to steal the gift of value v_1. If $x < v_1$, she either retains this gift or gets the next available gift with value less than x, depending on whether x is smaller or larger than g, the value of the gift currently held by player 1. In either case, if she opens a new gift, the final value she receives is no larger than v_1, so her best response is to steal the gift with value v_1. (The argument for player 1 is the same for all rounds in the game, and we do not repeat it for this case with only one unopened gift).

Now assume that it is some player $j \neq 1$'s turn to play when there is more than one unopened gift, and the induction hypothesis holds for the remainder of the game. Again, consider the case where 1 has not yet stolen a gift, so that her gift is still available. Suppose the values of the available gifts are $v_1 \geq v_2 \dots$ and so on. If j steals, she gets a final value of v_1. If she opens a new gift of value x, this gift becomes available and can be stolen in the remainder of the game. If $x > v_1$, by the induction hypothesis x is immediately stolen by the next player and j steals v_1, for a final value of v_1. If $x \leq v_1$, there is a sequence of stealing $v_1, v_2, \dots$; either j's gift of value x is never stolen in the remainder of the game, in which case her final value is $x \leq v_1$, or it is stolen. Irrespective of when it is stolen— either when player 1 has not yet stolen a gift, or after 1 steals a gift— j's final value is no larger than x: suppose x is stolen when 1 has not yet stolen a gift; by the induction hypothesis, j must steal the highest value available gift, which has value $v(x) \leq x$. (Since x was stolen, it was the highest value available gift at that point. Also note that such a gift definitely exists since 1's gift is available.) If j's gift is stolen after player 1 steals a gift and there are i' unopened gifts at this time, we must have $x \geq \theta_{i'}$ since x was stolen, because all players are playing according to the optimal strategy in G. By Theorem 1, j's expected utility from playing her optimal strategy at this point is $\max(v(x), \theta_{i'}) \leq x \leq v_1$. Therefore, player j can never get expected utility better than v_1, so she should steal the highest value available gift.

For player 1, when there are 2 or more unopened gifts, she can either steal the highest value available gift to obtain utility v_1, or open a new gift of value x. If she does not steal a gift now and never steals a gift in later rounds, by the induction hypothesis, every remaining gift including the last one is opened by 1 (since other players will steal the highest value available gift and 1's gift is always available). Also, by the induction hypothesis, if she steals a gift after this it has value greater than 1/2. In either case, she can ensure a expected utility of at least 1/2, so she should not steal if $v_1 \leq 1/2$. If $v_1 > 1/2$, then by opening the new gift of value x, the maximum value she can hope to get is $\max(v(x), 1/2)$, where $v(x)$ is the value of the best available gift after x. Since $v(x) \leq v_1$, her optimal strategy is to steal v_1 if $v_1 > 1/2$.

With these strategies, the optimal play of the game proceeds as follows: in the first step, player 1 opens a gift, which is stolen immediately by player 2; since the available set is empty, 1 opens a new gift, which is stolen immediately by 3, and so on; finally, the $n-1$th opened gift is stolen from 1 by player n. At this point, there are no available gifts, so 1 opens and keeps the last unopened gift; each gift, and therefore every player has expected value 1/2.

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